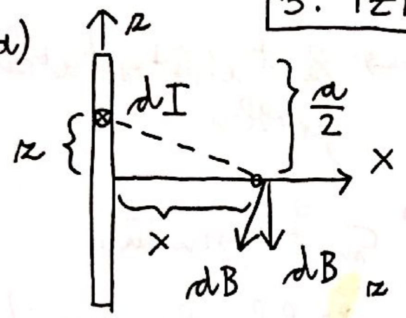
3. IZPIT IZ VAJ – REŠITVE NALOG

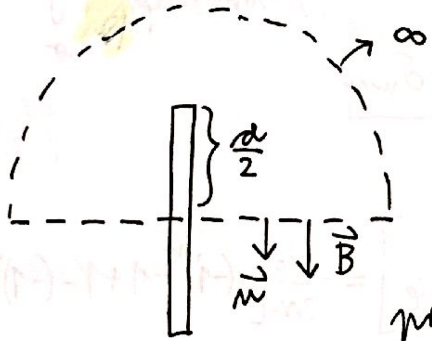
$\boxed{1}$ a)



$\boxed{+}$ $\begin{cases} dI = I \dfrac{dr}{a} \\[2ex] dB = \dfrac{\mu_0\, dI}{2\pi\sqrt{x^2+r^2}} \quad \text{polje vodnika} \end{cases}$

$\boxed{+}$ $\left\{ dB_r = \dfrac{x}{\sqrt{x^2+r^2}}\, dB = \dfrac{\mu_0 I}{2\pi a}\, \dfrac{x\, dr}{x^2+r^2} \right.$

$\frac{3}{4}$ $\boxed{+}$ kaže v smeri r navzdol $\leftarrow$ $B_r = \dfrac{\mu_0 I}{2\pi a} \displaystyle\int_{-\frac{a}{2x}}^{\frac{a}{2x}} \dfrac{d\left(\frac{r}{x}\right)}{1+\left(\frac{r}{x}\right)^2} = \boxed{\dfrac{\mu_0 I}{2\pi a}\, 2 \operatorname{arctg} \dfrac{a}{2x}}$ $\boxed{+}$



$x \gg a: \quad \operatorname{arctg} \dfrac{a}{2x} \approx \dfrac{a}{2x}, \quad \underbrace{B_r = \dfrac{\mu_0 I}{2\pi x}}_{\text{vodnik}}$ $\boxed{+}$

$x \ll a: \quad \operatorname{arctg} \dfrac{a}{2x} \approx \dfrac{\pi}{2}, \quad \underbrace{B_r = \dfrac{\mu_0 I}{2a}}_{\text{homogeno polje}}$ $\boxed{+}$

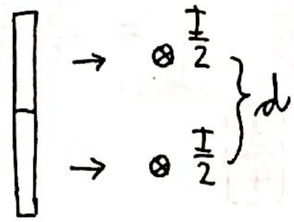
prispeva le integral po spodnji ploskvi:

b) $\left.\vec{F} = \dfrac{1}{\mu_0} \oint \left[\vec{B} \underbrace{\left(\vec{B}\cdot\vec{n}\right)}_{B} - \dfrac{1}{2} B^2 \vec{n} \right] dS = \dfrac{1}{\mu_0} \oint \dfrac{1}{2} B^2 \vec{n}\, \underbrace{dS}_{l\, dx} \right\}$ $\boxed{+}$

$\frac{1}{4}$ $\boxed{+}$ $\begin{cases} F = \dfrac{1}{2\mu_0}\, \dfrac{4\mu_0^2 I^2}{4\pi^2 a^2}\, l \displaystyle\int_{-\infty}^{\infty} \operatorname{arctg}^2 \dfrac{a}{2x}\, dx \\[3ex] \dfrac{F}{l} = \dfrac{\mu_0 I^2}{2\pi^2 a^2}\, 2a \underbrace{\displaystyle\int_{0}^{\infty} \operatorname{arctg}^2 \dfrac{a}{2x}\, d\left(\dfrac{x}{a}\right)}_{\frac{1}{2}\pi \ln 2} = \boxed{\dfrac{\mu_0 I^2}{2\pi a} \ln 2} \end{cases}$

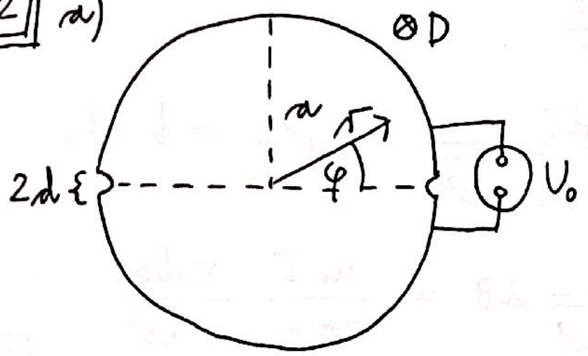
$\hookrightarrow$ tudi ta izraz že šteje za končni rezultat!

nadomestitev z vzporednima vodnikoma:



$$\frac{F}{l} = \frac{\mu_0 \left(\frac{I}{2}\right)\cdot\left(\frac{I}{2}\right)}{2\pi \frac{a}{4\ln 2}} \Rightarrow \underbrace{d = \frac{a}{4\ln 2}}$$

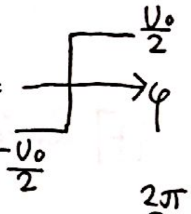
$\boxed{2}$ a)



$\nabla^2 U = 0 \rightarrow$ kot elektrostatski problem

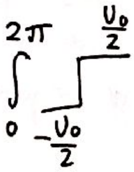
$\boxed{+}$ $U(r,\varphi) = \sum_m C_m r^m \sin m\varphi$

ostalih členov ni!

robni pogoj: $U(a,\varphi) = \sum_m C_m a^m \sin m\varphi =$ 

$/\cdot \sin n\varphi \rightarrow \int_0^{2\pi}$

$\boxed{+}$ $$\int_0^{2\pi} \sin m\varphi \sin n\varphi \, d\varphi = \delta_{mn} \underbrace{\int_0^{2\pi} \sin^2 m\varphi \, d\varphi}_{\frac{1}{2}\cdot 2\pi} = \underline{\pi \delta_{mn}}$$

$\int_0^{2\pi}$ 

$$\sin n\varphi \, d\varphi = -\frac{V_0}{2}\left[\underbrace{\int_0^{\pi} \sin n\varphi \, d\varphi}_{\frac{1}{n}(-\cos n\varphi)\big|_0^{\pi}} - \underbrace{\int_{\pi}^{2\pi} \sin n\varphi \, d\varphi}_{\frac{1}{n}(-\cos n\varphi)\big|_{\pi}^{2\pi}}\right] = -\frac{V_0}{2n}\left[-(-1)^n + 1 + 1 - (-1)^n\right] = \underline{-\frac{V_0}{n}\left[1-(-1)^n\right]}$$

$\Downarrow$

$\boxed{+}$ $$C_n a^n \pi = -\frac{V_0}{n}\left[1-(-1)^n\right] \Rightarrow \underline{C_n = -\frac{V_0}{\pi n}\left[1-(-1)^n\right]\frac{1}{a^n}}$$

$$U(r,\varphi) = -V_0 \sum_m \frac{1}{\pi m}\left[1-(-1)^m\right]\left(\frac{r}{a}\right)^m \sin m\varphi$$

zveznica med vdolbinama: $\boxed{+}$ $\underline{E = -\frac{1}{r}\frac{\partial U}{\partial \varphi}\Big|_{\varphi=0}}$ navpično iz simetrijskih razlogov

$$E = \frac{V_0}{r}\sum_m \frac{1}{\pi m}\left[1-(-1)^m\right]\left(\frac{r}{a}\right)^m m \underbrace{\cos m\varphi\big|_{\varphi=0}}_{1} =$$

$$= \frac{V_0}{r}\frac{1}{\pi}\sum_m \left[1-(-1)^m\right]\left(\frac{r}{a}\right)^m = \frac{V_0}{\pi r}\, 2\left[\underbrace{\frac{r}{a} + \left(\frac{r}{a}\right)^3 + \dots}_{\frac{r}{a}\cdot\frac{1}{1-\left(\frac{r}{a}\right)^2}}\right]$$ $\boxed{+}$

$\frac{3}{4}$

$$E = \frac{2V_0}{\pi r}\frac{r}{a}\cdot\frac{1}{1-\left(\frac{r}{a}\right)^2} = \boxed{\frac{2V_0}{\pi a}\cdot\frac{1}{1-\left(\frac{r}{a}\right)^2}}$$ $\boxed{+}$

b) $\vec{j} = \sigma \vec{E}$, $j = \sigma E$ v navpični smeri

$$I = \int_{-a+d}^{a-d} j\, D\, dr = \sigma D \int_{-a+d}^{a-d} E\, dr = \sigma D \frac{2U_0}{\pi} \underbrace{\int_{-a+d}^{a-d} \frac{d\left(\frac{r}{a}\right)}{1-\left(\frac{r}{a}\right)^2}}_{\int_{-1+\frac{d}{a}}^{1-\frac{d}{a}} \frac{dx}{1-x^2}}$$ $\boxed{+}$

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right] , \quad \int \frac{dx}{1-x^2} = \frac{1}{2} \ln \frac{1+x}{1-x}$$ $\boxed{\frac{1}{4}}$

$$\frac{1}{2} \ln \frac{1+x}{1-x} \Bigg|_{-1+\frac{d}{a}}^{1-\frac{d}{a}} = \frac{1}{2} \ln \frac{\boxed{2-\frac{d}{a}}^{\approx 2}}{\frac{d}{a}} - \frac{1}{2} \ln \frac{\frac{d}{a}}{2-\frac{d}{a}} \approx$$

$$\approx \frac{1}{2} \ln \frac{2a}{d} - \frac{1}{2} \ln \frac{d}{2a} = \underline{\ln \frac{2a}{d}}$$ $\boxed{+}$

$$I = \sigma D \frac{2U_0}{\pi} \ln \frac{2a}{d} = \frac{U_0}{R} \Rightarrow \boxed{R = \frac{\pi}{2\sigma D \ln \frac{2a}{d}}}$$

$\boxed{3}$ splošna rešitev valovne enačbe:

$\boxed{+}$ $$\{E_z, H_z\}(r,\varphi) = \sum_m \underbrace{J_m(\varkappa_{mn} r)}_{R(r)} \underbrace{(A_m \cos m\varphi + B_m \sin m\varphi)}_{\Phi(\varphi)}$$

$\omega(k_z) = c\sqrt{k_z^2 + \varkappa_{mn}^2}$ disperzijska relacija za posamezne načine

$\underline{\omega_{min} = c\, \varkappa_{mn}}$

- $\underline{\text{TM način}}$: $H_z = 0$, $E_z \neq 0$, robni pogoj $E_z = 0$

(1) brez pregrad: $\Phi(0) = \Phi(2\pi) \Rightarrow A_m, B_m$ oba lahko $\neq 0$

$\underline{m = 0, 1, 2, \dots}$

$\boxed{+}$ $\underline{\omega_{min1} = c\frac{\xi_{01}}{a}}$ $\quad E_z(r=a) = 0 \Rightarrow J_m(\varkappa_{mn} a) = 0$

$\varkappa_{mn} a = \xi_{mn}$

(2) s pregradami: $\Phi(0) = 0$ & $\Phi\left(\frac{2\pi}{3}\right) = 0$ $\quad m = 0 \rightarrow$ ničelno polje!

$\Downarrow \qquad \Downarrow$

$\boxed{+}$ $\underline{\omega_{min2} = c\frac{\xi_{\frac{3}{2}1}}{a}}$ $\quad A_m = 0 \qquad \sin\frac{2\pi m}{3} = 0 \Rightarrow \underline{m = \frac{3}{2}, 3, \frac{9}{2}, \dots}$

$\frac{1}{2}$ sprememba minimalne frekvence:

$\boxed{+}$ $$\frac{\omega_{min2}}{\omega_{min1}} = \frac{\xi_{\frac{3}{2}1}}{\xi_{01}} = \frac{4.5}{2.4} = \boxed{1.9}$$

- TE način: $E_z = 0$, $H_z \neq 0$, robni pogoj

$\boxed{+}$
$$\frac{\partial H_z}{\partial r} = 0 \quad \text{na obodu}$$
$$\frac{\partial H_z}{\partial \varphi} = 0 \quad \text{na pregradi}$$

$\boxed{+}$ 1) brez pregrad: $\Phi(0) = \Phi(2\pi) \Rightarrow A_m, B_m$ lahko oba $\neq 0$

$m = 0, 1, 2, \dots$

$$\frac{\partial H_z}{\partial \varphi}(r=a) = 0 \Rightarrow J_m'(\varkappa_{mn} a) = 0$$
$$\varkappa_{mn} a = \xi'_{mn}$$

$m = 0$ ne pride v poštev, saj je $\xi'_{01} = 0$

$\frac{1}{2}$ $$\omega_{min1} = c\,\frac{\xi'_{11}}{a}$$

$\boxed{+}$ 2) s pregradami: $\frac{\partial H_z}{\partial \varphi}(r, \varphi = 0) = 0$ & $\frac{\partial H_z}{\partial \varphi}(r, \varphi = \frac{2\pi}{3}) = 0$

$$\Downarrow \qquad\qquad \Downarrow$$
$$\Phi'(0) = 0 \qquad\qquad \Phi'\left(\frac{2\pi}{3}\right) = 0$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$B_m = 0 \qquad\qquad \sin\frac{2\pi m}{3} = 0, \quad m = 0, \frac{3}{2}, 3, \dots$$

$$\frac{\partial H_z}{\partial \varphi}(r=a) = 0 \Rightarrow J_m'(\varkappa_{mn} a) = 0$$
$$\varkappa_{mn} a = \xi'_{mn}$$

$m = 0$ spet ne pride v poštev

$$\omega_{min2} = c\,\frac{\xi'_{\frac{3}{2}1}}{a}$$

$\boxed{+}$ sprememba minimalne frekvence:

$$\frac{\omega_{min2}}{\omega_{min1}} = \frac{\xi'_{\frac{3}{2}1}}{\xi'_{11}} = \frac{2.6}{1.8} = \boxed{1.4}$$