Elektromagnetno polje: 1. vaje

(4. in 5. 10. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

0. Uvertura in ponovitev nekaj matematičnih pripomočkov

[*vektorska analiza (Gaussov izrek, gradient, divergenca), Fourierjeva transformacija, delta funkcija, debata o letošnji Nobelovi nagradi*]

1. Poissonova enačba za točkast naboj

[*Fourierjeva transformacija, Greenove funkcije*]

Reši Poissonovo enačbo za potencial električnega polja točkastega naboja e,

$$\nabla^2 U(\vec{r}) = -\frac{e}{\varepsilon_0}\delta(\vec{r}),$$

s pomočjo Fourierjeve transformacije.

Elektromagnetno polje: 2. vaje

(11. in 12. 10. 2016)

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1. Električno polje vodikovega atoma

[*določitev gostote naboje iz potenciala, odvajanje funkcije*]

Potencial električnega polja vodikovega atoma v osnovnem stanju je podan kot

$$U(r) = \frac{e}{4\pi\varepsilon_0}\frac{\mathrm{e}^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right),$$

kjer je r oddaljenost od jedra atoma in $\alpha^{-1} = a_B/2$ (a_B je Bohrov radij). Določi prostorsko gostoto naboja, ki vodi do takšnega potenciala. Kvalitativno interpretiraj dobljeni rezultat.

2. Električno polje nabite okrogle plošče

[*seštevanje prispevkov točkastih nabojev*]

Izračunaj jakost električnega polja vzdolž osi enakomerno nabite okrogle plošče s polmerom a, in sicer kot funkcijo oddaljenosti od plošče z. Površinska gostota naboja na plošči je σ. Končni rezultat poenostavi za dva posebna primera:

a) zelo blizu plošče in

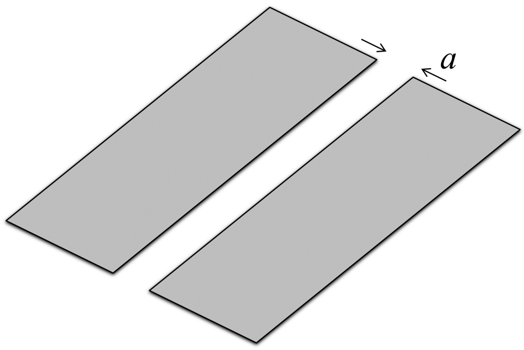
b) daleč stran od plošče.

Ustrezna rezultata primerjaj a) s poljem neskončne ravne plošče oziroma b) s poljem točkastega naboja.

3. Električno polje nabite ravne plošče z režo

[*seštevanje prispevkov točkastih nabojev*]

Iz velike tanke izolatorske plošče, ki je enakomerno nabita z nabojem površinske gostote σ, izrežemo ravno režo širine a, kakor prikazuje slika.



a) Določi jakost električnega polja E v ravnini, ki je pravokotna na ploščo in poteka skozi sredino reže, kot funkcijo oddaljenosti z od ravnine plošče ter podanih parametrov σ in a.

b) Pod a) dobljeni izraz za $E(z)$ poenostavi v limiti majhnih in velikih z ter skiciraj odvisnost $E(z)$.

Uporaben razvoj za $x > 0$: $\mathrm{arctg}\left(\frac{1}{x}\right) = \frac{\pi}{2} - x + \cdots$

4. Prečni trak v ploščatem kondenzatorju

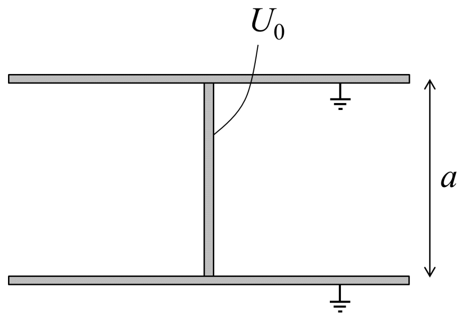
[*separacija spremenljivk v kartezičnih koordinatah*]

Med dve veliki vzporedni ravni prevodni plošči, ki se nahajata v medsebojni razdalji a, vstavimo dolg raven prevodni trak širine a, tako da je pravokoten na plošči in se plošč ravno še ne dotika (glej sliko). Plošči ozemljimo, na trak pa priključimo napetost U_0.

a) Izračunaj potencial električnega polja povsod znotraj takšnega kondenzatorja.

b) Poenostavi dobljeni izraz za velike oddaljenosti od traku.

c) Izračunaj jakost električnega polja v simetrijski ravnini kondenzatorja, vzporedni z njegovima ploščama. Dobljeno vrsto seštej.



Uporabna vrsta: $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)\, e^{-nz} = \frac{1}{2\operatorname{ch} z}$.

Elektromagnetno polje: 3. vaje

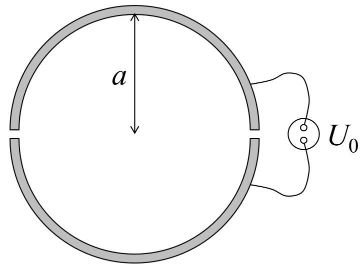
(18. in 19. 10. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Prepolovljena prevodna cev

[*separacija spremenljivk v valjnih koordinatah*]

Dolgo prevodno cev polmera a vzdolž osi prepolovimo, polovici rahlo razmaknemo in mednju priključimo konstantno napetost U_0, kakor v prečnem preseku cevi prikazuje slika. Stena cevi je tanka, razmik med polovicama cevi pa majhen v primerjavi z a.



a) Določi potencial električnega polja povsod *znotraj* cevi kot funkcijo valjnih koordinat r in φ (merjen od vodoravne ravnine) ter podanih parametrov a in U_0. Rezultat zapiši kot neskončno vrsto.

b) Pokaži, da jakost električnega polja v *vodoravni* simetrijski ravnini znotraj cevi kaže navpično navzdol in ima velikost

$$E(r) = \frac{2U_0 a}{\pi(a^2 - r^2)},$$

kjer je r oddaljenost od osi cevi.

c) Pokaži, da v *navpični* simetrijski ravnini znotraj cevi jakost električnega polja tudi kaže navpično navzdol, njena velikost pa je

$$E(r) = \frac{2U_0 a}{\pi(a^2 + r^2)}.$$

2. Prevodna krogla v homogenem električnem polju

[*separacija spremenljivk v krogelnih koordinatah, Legendrovi polinomi*]

Prevodno kroglo s polmerom a postavimo v zunanje homogeno električno polje z jakostjo E_0, s čimer se polje popači.

a) Izračunaj potencial nastalega električnega polja povsod v prostoru. Kvalitativno interpretiraj končni rezultat oziroma razloži obliko obeh dobljenih členov.

b) Izračunaj površinsko gostoto naboja, ki se inducira na površini krogle, kot funkcijo polarnega kota ϑ, merjenega od smeri zunanjega električnega polja.

c) Izračunaj električni dipolni moment induciranega naboja? Rezultat lahko prebereš naravnost iz rešitve pod a).

Elektromagnetno polje: 4. vaje

(25. in 26. 10. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Točkasti električni dipol v središču prevodne sfere

[*separacija spremenljivk v krogelnih koordinatah, Legendrovi polinomi*]

V središče izolirane votle prevodne sfere polmera a postavimo točkasti električni dipol z električnim dipolnim momentom p.

a) Določi potencial električnega polja povsod znotraj sfere.

b) Pokaži, da je električno polje naboja, ki se inducira na notranji površini sfere, *homogeno* in izračunaj njegovo velikost?

c) Izračunaj skupni dipolni moment induciranega naboja. Kakšno smer ima glede na točkasti dipol? Je rezultat presenetljiv?

2. Točkasti naboj nad prevodno ploščo

[*zrcaljenje*]

V razdalji d nad veliko ozemljeno prevodno ploščo se nahaja točkasti naboj e.

a) Določi potencial električnega polja povsod v prostoru. Kakšno je električno polje pod ploščo, se pravi na drugi strani?

b) Izračunaj površinsko gostoto naboja, ki se inducira na plošči, kot funkcijo oddaljenosti r od točke na plošči, ki je najbližje točkastemu naboju. Pokaži, da celotni induciran naboj na plošči znaša ravno $-e$. Ali lahko do tega rezultata prideš tudi na enostaven način?

Elektromagnetno polje: 5. vaje
(2. 11. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Točkasti naboj nad prevodno kroglo
[*zrcaljenje*]

V razdalji D od središča prevodne ozemljene krogle polmera a se nahaja točkasti naboj e.

a) Pokaži, da je potencial električnega polja izven krogle enak, kot če bi kroglo nadomestili z drugim točkastim nabojem, ki bi ga postavili na zveznico med prvim točkastim nabojem in središčem krogle. Določi velikost tega naboja in mesto, na katerega bi ga morali postaviti. Kolikšen je celotni naboj, ki se inducira na površini krogle?

b) V primeru, da krogla ni ozemljena, je potencial električnega polja enak, kot če bi kroglo nadomestili z dvema točkastima nabojema namesto z enim samim. Pokaži, da moramo drugega postaviti v središče krogle in da mora biti nasprotno enak prvemu.

Pri obeh vprašanjih se zavedaj, da je površina krogle ekvipotencialna ploskev.

2. Točkasti naboj v kotu med dvema pravokotnima prevodnima ploščama
[*zrcaljenje, multipolni razvoj*]

Točkasti naboj e se nahaja v kotu med dvema razsežnima prevodnima ozemljenima ploščama, ki sta pravokotni druga na drugo, tako da je od vsake oddaljen za razdaljo a.

a) Izračunaj kvadrupolni moment nastale porazdelitve nabojev.

b) Kako se obnaša potencial električnega polja v veliki oddaljenosti, $r \gg a$?

Potencial električnega polja, ki ga povzroči lokalizirana porazdelitev nabojev v točki $\vec{r}$, v multipolnem razvoju zapišemo kot

$$U(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{r} + \sum_i p_i \frac{r_i}{r^3} + \sum_{ij} Q_{ij}\frac{r_i r_j}{r^5} + \cdots\right),$$

kjer so $p_i = \int \rho(\vec{r}\,')r_i'\,\mathrm{d}^3\vec{r}\,'$ komponente vektorja dipolnega momenta in

$$Q_{ij} = \int \rho(\vec{r}\,')\left[3r_i' r_j' - \delta_{ij} r'^2\right]\mathrm{d}^3\vec{r}\,'$$

komponente tenzorja kvadrupolnega momenta, $\rho(\vec{r}\,')$ pa je prostorninska gostota naboja v točki $\vec{r}\,'$.

3. Električna sila na točkasti naboj nad prevodno ploščo
[*napetostni tenzor električnega polja*]

V razdalji d nad veliko ozemljeno prevodno ploščo se nahaja točkasti naboj e. Z uporabo napetostnega tenzorja izračunaj električno silo na točkasti naboj. Rezultat primerjaj s silo med točkastima nabojema e in $-e$ v medsebojni razdalji $2d$.

Elektromagnetno polje: 6. vaje

(8. in 9. 11. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Električna sila na polovico prevodne krogle

[*napetostni tenzor električnega polja*]

Prevodno kroglo polmera a postavimo v navpično homogeno električno polje jakosti E_0. Izračunaj električno silo, ki deluje na zgornjo polovico krogle. V katero smer kaže ta sila?

2. Magnetno polje tokovne zanke

[*vektorski potencial magnetnega polja, magnetni dipolni moment*]

Izračunaj vektorski potencial magnetnega polja krožne zanke s polmerom a in električnim tokom I v veliki oddaljenosti od zanke. Rezultat podaj z oddaljenostjo r od središča zanke in s kotom ϑ glede na os zanke. Pri računu obdrži le vodilni člen v razvoju po r. Pokaži, da ima tako dobljeni rezultat obliko vektorskega potenciala magnetnega dipola z magnetnim dipolnim momentom $\pi a^2 I$.

3. Magnetno polje nabite vrteče se okrogle plošče

[*Biot-Savartova enačba, magnetni dipolni moment*]

Tanko okroglo ploščo polmera a enakomerno premažemo z nabojem površinske gostote σ in jo v vodorvanem položaju zavrtimo z enakomerno kotno hitrostjo ω okrog navpične osi, ki poteka skozi središče plošče.

a) Z uporabo Biot-Savartove enačbe izračunaj velikost gostote magnetnega polja B na navpični osi plošče kot funkcijo oddaljenosti z od središča plošče.

b) Pokaži, da je magnetni dipolni moment plošče $p_m = \frac{\pi}{4}\sigma\omega a^4$.

c) V razvoju pod a) izračunanega izraza za $B(z)$ v Taylorjevo vrsto določi člen, ki najpočasneje pada z z. Utemelji, zakaj je to dipolni člen. Iz njegove oblike preberi magnetni dipolni moment plošče in ga primerjaj z izrazom pod b).

Elektromagnetno polje: 7. vaje

(15. in 16. 11. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Magnetna sila v koaksialnem kablu

[*napetostni tenzor magnetnega polja*]

Dolg koaksialni kabel je sestavljen iz tanke prevodne cevi polmera a, po osi katere poteka tanek prevodni vodnik. Po vodniku spustimo električni tok I, ki se v nasprotni smeri vrača enakomerno porazdeljen po cevi. Izračunaj silo na dolžinsko enoto, s katero je po obodu napeta cev koaksialnega kabla.

2. Magnetna sila v dolgi tuljavi

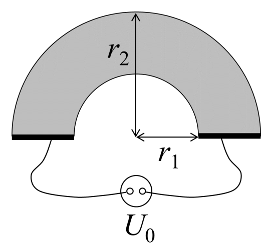
[*napetostni tenzor magnetnega polja*]

Po dolgi ravni tuljavi dolžine l s številom ovojev N, ki imajo polmer po a, teče električni tok I. Izračunaj silo, s katero je po obodu napet posamezni ovoj tuljave.

3. Upor prevodne ploščice

[*potencial električnega polja v prevodniku*]

Iz kovinske plošče debeline d s specifično prevodnostjo σ izrežemo ploščico v obliki polovice kolobarja z notranjim polmerom r_1 in zunanjim polmerom r_2. Na ravni stranici ploščice naparimo elektrodi iz zelo dobrega prevodnika, mednju pa priključimo izvor konstantne napetosti U_0, kakor prikazuje slika. Določi potencial električnega polja v ploščici in s pomočjo tega izračunaj upor ploščice.



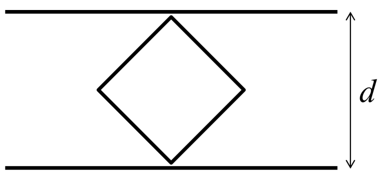
4. Indukcija v okvirju

[*indukcija, lastna induktivnost, medsebojna induktivnost*]

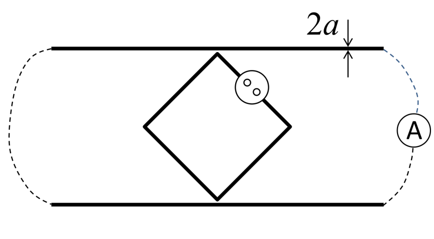
Kvadratni okvir iz tankega vodnika postavimo med dva vzporedna dolga tanka ravna vodnika, tako da ravnina okvirja sovpada z ravnino, ki jo definirata vodnika, diagonala okvirja je pravokotna na vodnika, skrajni točki okvirja pa se ravno še ne dotikata vodnikov (glej prvo sliko). Razdalja med dolgima vodnikoma in dolžina diagonale okvirja znašata po d.

a) Pokaži, da je medsebojna induktivnost okvirja in para vodnikov

$$L_{12} = \frac{2\ln 2}{\pi}\mu_0 d.$$



b) Okvir napajamo z izmeničnim tokom amplitude I_1. Kakšna je amplituda toka I_2, ki se inducira v vzporednih vodnikih, če ju *daleč stran* sklenemo (druga slika)? Pri tem delu naloge upoštevaj, da imata vodnika debelino $2a$ in dolžino l, tako da je $a \ll d$ in $l \gg d$. Rezultat za I_2/I_1 izrazi s parametri d, a in l ter ga numerično izvrednoti za $l/d = d/a = 10$.



Elektromagnetno polje: 8. vaje

(22. in 23. 11. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Cabrerin eksperiment

[*magnetni monopoli, indukcija*]

Blas Cabrera je leta 1982 poročal o eksperimentu, v katerem je v 151 dneh opazovanja domnevno zaznal magnetni monopol. Za zaznavo magnetnega monopola je uporabil krožno kovinsko zanko v superprevodnem stanju, skozi katero je meril električni tok. Predpostavi, da magnetni monopol z magnetnim nabojem g potuje s hitrostjo v po osi takšne krožne zanke polmera a in induktivnosti L.

a) Izračunaj in nariši časovni potek magnetnega pretoka skozi zanko. Magnetno polje monopola je v točki $\vec{r}$ glede na monopol podano z enačbo

$$\vec{B}(\vec{r}) = \frac{\mu_0 g}{4\pi} \frac{\vec{r}}{r^3}.$$

b) Izračunaj in nariši časovni potek v zanki induciranega električnega toka. Posplošeni Faradayev zakon za primer obstoja magnetnih monopolov zapišemo kot

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{j}_m,$$

kjer je $\vec{j}_m$ vektor gostote toka magnetnih nabojev.

c) Iz rezultata pod b) sledi, da pri prečkanju magnetnega monopola magnetni pretok skozi zanko skoči za vrednost $\mu_0 g$. Pokaži, da to ustreza ravno dvema kvantoma magnetnega pretoka h/e. Kvantizacijo magnetnega pretoka po Diracu zapišemo kot $\frac{1}{2}\mu_0 g e = h$.

Cabrerin eksperiment je zaznal natanko en magnetni monopol. Kasnejši podobni eksperimenti magnetnih monopolov niso več zaznali.

Elektromagnetno polje: 9. vaje

(29. in 30. 11. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Izmenični tok po valjastem vodniku

[*kvazistatična aproksimacija Maxwellovih enačb*]

Med konca valjastega vodnika iz kovine s specifično prevodnostjo σ priključimo izvor izmenične napetosti amplitude U_0 in krožne frekvence ω. Polmer vodnika je a, njegova dolžina pa l.

a) Izračunaj radialno odvisnost amplitud jakosti električnega polja in gostote magnetnega polja v vodniku.

b) Na podlagi rezultata pod a) izračunaj radialno porazdelitev amplitude gostote električnega toka in jo poenostavi v limitah majhnih in velikih ω. Pokaži, da je v limiti velikih ω električni tok zgoščen na površini vodnika.

c) Izračunaj impedanco vodnika. Rezultat izrazi z uporom vodnika $R_0 = l/(\sigma\pi a^2)$ in ga poenostavi v limitah majhnih in velikih ω.

Računaj v kvazistatični aproksimaciji.

2. Poprava prvega kolokvija

Elektromagnetno polje: 10. vaje

(6. in 7. 12. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Energijski tok v koaksialnem kablu in v valjastem vodniku

[*Poyntingov izrek*]

Izračunaj energijski tok skozi prečni presek oziroma skozi zunanjo površino naslednjih dveh vodnikov:

a) koaksialnega kabla, kjer je napetost med žilo in plaščem U, ta pa po njima v nasprotnih smereh poganja električni tok I,

b) dolgega ravnega vodnika preseka S in dolžine l iz kovine s specifično prevodnostjo σ, po katerem teče električni tok I. Končni rezultat izrazi s celotnim uporom vodnika $R = l/(\sigma S)$.

V obeh primerih interpretiraj dobljeni rezultat.

2. Prekinjeni vodnik

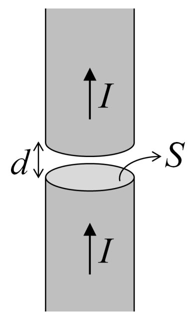
[*Poyntingov izrek*]

Dolg raven valjasti vodnik preseka S je na nekem mestu prekinjen. Prekinitev ima obliko ozke špranje širine d pravokotne na vodnik (glej sliko). Ob času $t = 0$ po vodniku spustimo konstanten električni tok I, zaradi katerega se na zgornji in spodnji meji špranje začne nabirati naboj.

a) Določi smer in velikost jakosti električnega polja ter gostote magnetnega polja v špranji v oddaljenosti r od osi vodnika ob času t.

b) S pomočjo Poyntingovega vektorja izračunaj energijski tok, ki ob času t priteka v špranjo.

c) Prejšnji rezultat primerjaj s časovnim odvodom energije elektromagnetnega polja v špranji.

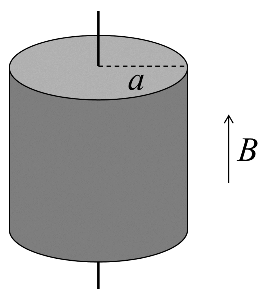


Pri vseh računih zanemari popačitev polj ob zunanjem robu špranje. Špranjo torej obravnavaj kot ploščati kondenzator. Upor vodnika zanemari.

3. Feynmanov paradoks v valjni različici

[*vrtilna količina elektromagnetnega polja*]

Dolg raven vodnik je enakomerno nabit z nabojem dolžinske gostote $-\lambda$. Vodnik je obdan z dolgim izolatorskim valjem, ki se lahko prosto vrti okoli svoje osi, ki sovpada z vodnikom (glej sliko). Vztrajnostni moment valja na dolžinsko enoto je j, površina valja pa je premazana z nabojem površinske gostote $\lambda/(2\pi a)$, kjer je a polmer valja, tako da je skupni naboj ravno nasproten skupnemu naboju na vodniku. Sprva je v prostoru homogeno magnetno polje B v smeri vodnika, ki ga nato počasi ugasnemo.



a) Preko spremembe vrtilne količine elektromagnetnega polja izračunaj, s kakšno kotno hitrostjo se valj zavrti.

b) Kotno hitrost izračunaj tudi neposredno preko Faradayevega zakona in se prepričaj, da dobiš enak rezultat.

c) Če dolžinsko gostoto naboja na ravnem vodniku spremenimo na vrednost $-\lambda' \neq -\lambda$, medtem ko površinska gostota naboja na valju ostane $\lambda/(2\pi a)$, se pod a) izračunana kotna hitrost *spremeni.* Tedaj je namreč na začetku tudi v prostoru okrog valja električno polje neničelno, s tem pa tudi vrtilna količina elektromagnetnega polja. Po drugi strani pa se pod b) izračunana kotna hitrost *ne spremeni.* To navidezno protislovje se imenuje Feynmanov paradoks. Kako ga razrešiti?

Elektromagnetno polje: 11. vaje

(13. in 14. 12. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Radialno polarizirana krogla

[*polarizacija, vezani naboj*]

Krogla polmera a je polarizirana tako, da ima vektor polarizacije znotraj krogle krajevno odvisnost $\vec{P}(\vec{r}) = k\vec{r}$, kjer je k znana konstanta. Izračunaj:

a) prostorninsko gostoto vezanega naboja v krogli, površinsko gostoto vezanega naboja na površini krogle in skupni naboj v krogli,

b) jakost električnega polja povsod po prostoru.

Rezultat pod b) pokaže, da je električno polje v krogli kar sorazmerno s polarizacijo. Zakaj?

2. Prepolovljena polarizirana krogla

[*polarizacija, vezani naboj*]

Kroglo, izdelano iz snovi s homogeno polarizacijo $\vec{P}$, prerežemo na pol, tako da gre rez skozi središče krogle in je pravokoten na $\vec{P}$. Obe polovici krogle malenkost razmaknemo, tako da je razmik *zelo majhen* v primerjavi s polmerom krogle. Izračunaj gostoto električnega polja v špranji med polovicama krogle. Najprej reši nalogo za neprerezano kroglo in razmisli, kako se rezultat spremeni v opisanem primeru.

3. Ploščica iz anizotropnega dielektrika

[*tenzor dielektrične konstante, robni pogoji za snov*]

Med plošči ploščatega kondenzatorja kapacitete C_0 vstavimo ploščico anizotropnega dielektrika, tako da ploščica zapolnjuje celotno prostornino kondenzatorja. Dielektrična konstanta ima lastne vrednosti ε_1, ε_1 in ε_2, ploščica pa je odrezana tako, da lastna os, ki ji ustreza lastna vrednost ε_2, z normalo plošč oklepa kot φ. Izračunaj kapaciteto tako zapolnjenega kondenzatorja.

Elektromagnetno polje: 12. vaje

(20. in 21. 12. 2016)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Točkast električni dipol v krogelni votlini dielektrika

[*dielektrična konstanta, vezani naboj, robni pogoji za snov*]

V razsežni homogeni snovi z dielektrično konstanto ε je krogelna votlina polmera a. V njeno središče postavimo točkast električni dipol z električnim dipolnim momentom p.

a) Izračunaj potencial električnega polja povsod po prostoru kot funkcijo krogelnih koordinat r in ϑ. Na podlagi dobljenega izraza pokaži, da ima električno polje zunaj krogelne votline obliko polja električnega dipola z električnim dipolnim momentom $p' = \frac{3p}{2\varepsilon+1}$. Polarni kot ϑ je merjen od smeri dipola.

b) Izračunaj *površinsko* gostoto vezanega naboja na površini krogelne votline kot funkcijo polarnega kota ϑ. Izhajaš lahko iz pod a) podanega izraza za p'.

c) Utemelji, zakaj je *prostorninska* gostota vezanega naboja povsod v snovi enaka nič.

2. Elektromagnetni valovi v hladni plazmi

[*zveza med makroskopskimi in mikroskopskimi količinami*]

Pri obravnavi potovanja elektromagnetnih valov po hladni plazmi lahko predpostavimo, da sestavni ioni zaradi velike mase skoraj mirujejo, sestavni elektroni pa so skoraj povsem prosti, tako da se hitro odzivajo na zunanja polja. Ker je plazma hladna, lahko termično gibanje elekronov zanemarimo.

a) S pomočjo enačbe gibanja za proste elektrone mase m in naboja $-e$ pokaži, da frekvenčno odvisnost dielektričnosti plazme zapišemo kot $\varepsilon(\omega) = 1 - \omega_p^2/\omega^2$, kjer je $\omega_p = \sqrt{ne^2/(m\varepsilon_0)}$ plazemska frekvenca in n številska gostota elektronov v plazmi.

b) S pomočjo rezultata pod a) pokaži, da je disperzijska relacija elektromagnetnih valov, ki se lahko širijo po plazmi, $\omega(k) = \sqrt{\omega_p^2 + c_0^2 k^2}$, kjer je c_0 hitrost elektromagnetnega valovanja v vakuumu. Posebej obravnavaj limitna primera velikih in majhnih frekvenc.

c) S pomočjo rezultata pod b) izračunaj in skiciraj frekvenčno odvisnost fazne in grupne hitrosti elektromagnetnih valov v plazmi. Primerjaj obe hitrosti s hitrostjo svetlobe v vakuumu.

3. Longitudinalni elektromagnetni valovi v snovi

[*konstitutivna relacija*]

Za popoln opis obnašanja električnega polja v snovi moramo poznati dodatno zvezo med posameznimi polji in porazdelitvami, med katerimi so $\vec{E}$, $\vec{D}$, $\vec{P}$, $\vec{j}$ in ρ. Takšni zvezi pravimo konstitutivna relacija. V običajnih dielektrikih je to zveza med $\vec{D}$ in $\vec{E}$, ki definira dielektrično konstanto.

V neki snovi se konstitutivna relacija glasi

$$\frac{\partial \vec{j}}{\partial t} + c_0^2 \nabla \rho = \varepsilon_0 \omega_p^2 \vec{E},$$

kjer je $\vec{j}$ prostorninska gostota električnega toka, ρ prostorninska gostota naboja, c_0 in ω_p pa znani konstanti. Pokaži, da se po snovi lahko širijo *longitudinalni* valovi in določi njihovo disperzijsko relacijo. Ali, sodeč po dobljeni disperzijski relaciji, prepoznaš za kakšno snov gre?

Pojav longitudinalnih elektromagnetnih valov je redkejši kot pojav transverzalnih valov, kakršni so, denimo, edini možni v vakuumu.

Elektromagnetno polje: 13. vaje

(3. in 4. 1. 2017)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Valovni vodnik iz vzporednih prevodnih plošč

[*širjenje elektromagnetnega valovanja v omejeni geometriji*]

Veliki vzporedni prevodni plošči v medsebojni razdalji a uporabimo kot valovni vodnik.

a) Če smer širjenja valovanja označimo z z, pokaži, da lahko komponente E_x, E_y, H_x in H_y jakosti električnega in magnetnega polja vse izrazimo s komponentama E_z in H_z. Za popolno poznavanje elektromagnetnega polja v valovnem vodniku torej zadostuje poiskati krajevni odvisnosti vzdolžnih komponent E_z in H_z. To velja v splošnem, za valovni vodnik s poljubnim presekom.

b) Izračunaj krajevno odvisnost vzdolžne komponente E_z za transverzalni magnetni (TM) način valovanja, pri katerem je $H_z = 0$, in krajevno odvisnost vzdolžne komponente H_z za transverzalni električni (TE) način valovanja, pri katerem je $E_z = 0$. Za oba primera izračunaj tudi disperzijsko relacijo valovanja.

c) Pokaži, da ima v TM načinu impedanca valovnega vodnika, ki jo definiramo kot $Z = E_\perp/H_\parallel$ (kjer je $E_\perp$ komponenta električnega polja pravokotna na plošči, $H_\parallel$ pa komponenta magnetnega polja vzporedna s ploščama), frekvenčno odvisnost $Z = Z_0\sqrt{1-\omega_0^2/\omega^2}$, kjer je $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ upor vakuuma in ω_0 najnižja možna frekvenca valovanja v uporabljenem valovnem načinu.

d) Pokaži, da je v TM načinu razmerje *amplitud* prečne in vzdolžne komponente jakosti električnega polja enako k/κ, kjer je k valovni vektor valovanja in κ valovni vektor, ki opisuje prečno krajevno odvisnost polj. Ta rezultat nam omogoča preprosto predstavo širjenja valovanja vzdolž plošč: valovanje izgleda kot periodično odbijanje valovnih front med obema ploščama.

Elektromagnetno polje: 14. vaje

(10. in 11. 1. 2017)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Valjasta cev kot valovni vodnik

[*širjenje elektromagnetnega valovanja v omejeni geometriji*]

Dolgo prevodno cev polmera a uporabimo kot valovni vodnik.

a) Izračunaj krajevno odvisnost vzdolžne komponente jakosti električnega polja $E_z(r, \varphi)$ za transverzalni magnetni (TM) način valovanja in krajevno odvisnost vzdolžne komponente jakosti magnetnega polja $H_z(r, \varphi)$ za transverzalni električni (TE) način valovanja, kjer os cevi kaže vzdolž z, medtem ko sta r in φ valjni koordinati v ravnini pravokotni na z.

b) Za vsak način valovanja določi disperzijsko relacijo in izračunaj najmanjšo frekvenco, pri kateri se valovanje še lahko širi po vodniku.

Spodnji tabeli povzemata ničle Besslovih funkcij in odvodov Besslovih funkcij.

k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

k	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

2. Transverzalni električni in magnetni (TEM) valovi v valovnem vodniku

[*širjenje elektromagnetnega valovanja v omejeni geometriji*]

Pri transverzalnih električnih in magnetnih (TEM) valovih sta električno in magnetno polje, $\vec{E}$ in $\vec{H}$, pravokotni na smer širjenja valovanja. V praznem prostoru je to edini način širjenja valovanja, v valovnih vodnikih pa je to poseben način, ki obstaja le pod določenimi pogoji.

a) Pokaži, da v TEM načinu velja $\nabla \times \vec{E} = i\vec{k} \times \vec{E}$, kjer je $\vec{k}$ valovni vektor, in podobno za $\vec{H}$. S pomočjo teh dveh zvez pokaži, da je disperzijska relacija TEM valovanja linearna, $\omega = ck$, kjer je c hitrost valovanja.

b) Pokaži, da sta amplitudi obeh polj kar rešitvi Laplaceove enačbe, $\nabla_\perp^2 \vec{E} = 0$ in $\nabla_\perp^2 \vec{H} = 0$, kjer $\nabla_\perp$ označuje operator gradienta v smeri pravokotni na smer širjenja valovanja. Hkrati ti dve enačbi predstavljata statično limito valovne enačbe, torej limito $\omega = 0$ in $k = 0$. To pomeni, da je iskanje TEM načina valovanja ekvivalentno reševanju statičnega problema za dani valovni vodnik.

c) Na podlagi rezultata pod b) razloži, zakaj se TEM valovanje ne more širiti v valovnih vodnikih s sklenjenim presekom, lahko pa širi, na primer, v koaksialnem kablu ali med dvema vzporednima ploščama.

3. TEM valovanje v koaksialnem kablu

[*širjenje elektromagnetnega valovanja v omejeni geometriji*]

Koaksialni kabel je sestavljen iz dveh dolgih prevodnih cevi polmerov a in b ter tankih sten. Prostor med cevema je zapolnjen s snovjo, ki se obnaša kot plazma s frekvenčno odvisnostjo dielektrične konstante

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2},$$

kjer je ω_p plazemska frekvenca. V takšen valovni vodnik spustimo elektromagnetno valovanje v TEM načinu.

a) Izračunaj disperzijsko relacijo elektromagnetnega valovanja v valovnem vodniku.

b) Izračunaj frekvenčno odvisnost impedance vodnika in jo skiciraj. Pojasni, zakaj impedanca pri frekvenci ω_p divergira.

4. Sevanje kratke dipolne antene

[*sevalni približek*]

Ravni vodnik dolžine l, ki ga napajamo z izmeničnim tokom $I = I_0 \sin(\omega t)$, ki je povsod vzdolž vodnika *enak*, uporabimo kot oddajno anteno. Vodnik je kratek v primerjavi z valovno dolžino $\lambda = 2\pi c_0/\omega$ izsevanega valovanja, kjer je c_0 hitrost svetlobe v vakuumu. Takšni anteni pravimo tudi Hertzov dipol.

a) Izračunaj krajevno in časovno odvisnost magnetnega in električnega polja, $\vec{B}(\vec{r}, t)$ in $\vec{E}(\vec{r}, t)$, v sevalnem približku, torej daleč stran od antene.

b) S pomočjo Poyntingovega vektorja izračunaj časovno povprečje celotne moči izsevanega valovanja. Dobljeni rezultat zapiši kot ZI_{eff}^2, kjer je Z sevalni upor antene in $I_{\text{eff}} = I_0/\sqrt{2}$ efektivni tok v anteni, in na ta način izračunaj Z.

Elektromagnetno polje: 15. vaje

(17. in 18. 1. 2017)

asistent: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Sevanje dipolne antene

[*sevalni približek*]

Ko ravni vodnik uporabimo kot oddajno dipolno anteno, je tok po njegovi dolžini običajno porazdeljen. Vzdolž vodnika se namreč vzpostavi stoječi tokovni val, ki ga za vodnik dolžine l, napajan na sredini, v splošnem zapišemo kot

$$I(z', t) = I_0 \sin\left[k\left(\frac{l}{2} - |z'|\right)\right]\sin(\omega t),$$

kjer je I_0 amplituda toka, z' koordinata vzdolž vodnika (ki teče od $-l/2$ do $l/2$), k pa valovni vektor. Pri zapisani tokovni porazdelitvi tok na robovih vodnika pade na nič, kakor bi pričakovali. Stoječi tokovni val ima posebej lepo obliko, če je dolžina vodnika lihi večkratnik $\lambda/2$, se pravi $l = \lambda/2$, $3\lambda/2\ldots$

Za primera $l = \lambda/2$ in $l = 3\lambda/2$ izpelji in skiciraj prostorsko porazdelitev gostote energijskega toka, ki ga seva takšna dipolna antena. Rezultat za $l = \lambda/2$ primerjaj z rezultatom za preprostejši primer Hertzovega dipola, tako da oba narišeš na isto sliko.

2. Sevanje majhne tokovne zanke

[*sevalni približek*]

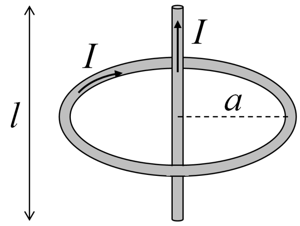
Krožno zanko polmera a, ki jo napajamo z izmeničnim tokom $I = I_0\sin(\omega t)$, uporabimo kot oddajno anteno. Zanka je majhna v primerjavi z valovno dolžino $\lambda = 2\pi c_0/\omega$ izsevanega valovanja, kjer je c_0 hitrost svetlobe v vakuumu.

a) Izračunaj krajevno in časovno odvisnost magnetnega in električnega polja, $\vec{B}(\vec{r}, t)$ in $\vec{E}(\vec{r}, t)$, v sevalnem približku, torej daleč stran od antene.

b) S pomočjo Poyntingovega vektorja izračunaj časovno povprečje celotne moči izsevanega valovanja. Dobljeni rezultat zapiši kot ZI_{eff}^2, kjer je Z sevalni upor antene in $I_{\text{eff}} = I_0/\sqrt{2}$ efektivni tok v anteni, in na ta način izračunaj Z.

3. Sevanje kombinirane antene
[*sevalni približek*]

Za oddajanje krožno polariziranega valovanja lahko uporabimo anteno v obliki nesklenjene vodoravne krožne zanke z navpičnima koncema (glej sliko). Predpostavi, da je antena *majhna* v primerjavi z valovno dolžino λ valovanja, ki ga oddaja. Potem jo lahko obravnavamo kot kombinacijo vodoravne krožne zanke polmera a in navpične prečke dolžine l, ki simetrično prebada zanko (glej sliko). Anteno napajamo z električnim tokom $I = I_0 \sin \omega t$.



a) Določi električni in magnetni dipolni moment takšne antene kot funkcijo časa t.

b) Pokaži, da je v poljubni smeri valovanje, ki ga takšna antena oddaja, eliptično polarizirano.

c) Kako moramo izbrati l pri danem a in dani valovni dolžini λ, da bo valovanje res krožno polarizirano?

V sevalnem približku je gostota magnetnega polja nihajočega električnega dipola $\vec{p}_e$ v točki $\vec{r}$ podana kot $\vec{B}_e = -\frac{\mu_0}{4\pi c_0 r}\, \hat{e}_r \times \ddot{\vec{p}}_e(t - \frac{r}{c_0})$, ustrezni rezultat za magnetni dipol $\vec{p}_m$ pa je $\vec{B}_m = -\frac{\mu_0}{4\pi c_0^2 r}\, \hat{e}_r \times \left[\hat{e}_r \times \ddot{\vec{p}}_m(t - \frac{r}{c_0})\right]$, kjer je c_0 hitrost svetlobe in $\hat{e}_r = \vec{r}/r$.

Electromagnetic field: 1st tutorial class

(4th in 5th of October 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

0. Introduction and recapitulation of some mathematical tools
[*vector analysis (Gaussian theorem, gradient, divergence), Fourier transform, delta function, discussion about this year's Nobel prize*]

1. Poisson equation for the point charge
[*Fourier transform, Green functions*]

Solve the Poisson equation for the electric field potential of the point charge e,

$$\nabla^2 U(\vec{r}) = -\frac{e}{\varepsilon_0}\delta(\vec{r}),$$

using the Fourier transform.

Electromagnetic field: 2nd tutorial class

(11th in 12th of October 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Electric field of the hydrogen atom

[*determination of the charge density from the potential, function derivative*]

The electric field potential of the hydrogen atom in its ground state is given as

$$U(r) = \frac{e}{4\pi\varepsilon_0}\frac{\mathrm{e}^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right),$$

where r is the distance from the atomic nucleus and $\alpha^{-1} = a_B/2$ (a_B is the Bohr radius). Determine the volume charge density leading to such a potential. Interpret the obtained result.

2. Electric field of a charged circular plate

[*summing the contributions of the point charges*]

Determine the electric field strength along the axis of a uniformly charged circular plate of radius a as a function of the distance z from the plate. The surface density of the charge on a plate amounts to σ. Simplify the final result in two special cases:

a) near the plate and

b) far away from the plate.

Compare the obtained results a) with the field of an infinite charged plane and b) with the field of a point charge.

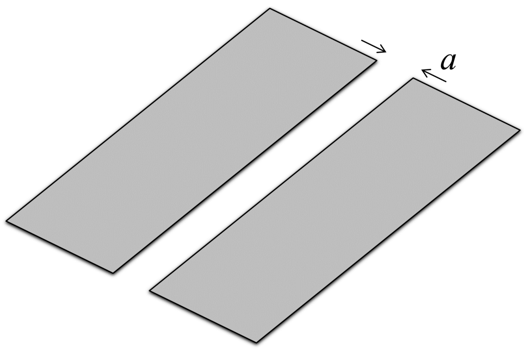
3. Electric field of a charged plane with a slit

[*summing the contributions of the point charges*]

A large flat plate uniformly charged with the surface charge density σ exhibits a straight slit of width a, as shown in the figure.

a) Determine the electric field strength E in the plane perpendicular to the plate, which runs through the middle of the slit, as a function of the distance z from the plate.

b) Simplify the obtained result for $E(z)$ in the limits of small and large z and plot $E(z)$.



Useful series expansion for $x > 0$: $\mathrm{arctg}\left(\frac{1}{x}\right) = \frac{\pi}{2} - x + \cdots$

4. Transverse ribbon in a parallel plate capacitor

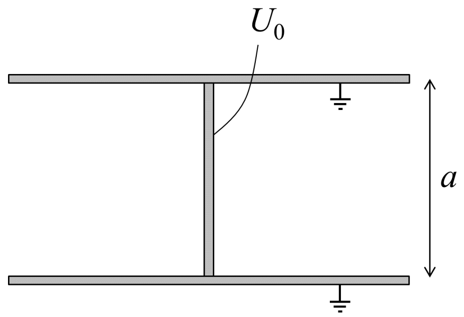
[*separation of variables in cartesian coordinates*]

A long flat conducting ribbon of width a is placed in between the two parallel flat conducting plates separated by a, so that the ribbon is perpendicular to the plates and does not touch them (see figure). The plates are grounded and the ribbon is held at the potential U_0.

a) Determine the electric field potential everywhere inside such a capacitor.

b) Simplify the obtained result for large distances from the ribbon.

c) Calculate the electric field strength in the symmetry plane of the capacitor parallel to the plates. Sum the obtained series.



Useful series expansion: $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)\, e^{-nz} = \frac{1}{2\,\mathrm{ch}\, z}$.

Electromagnetic field: 3rd tutorial class

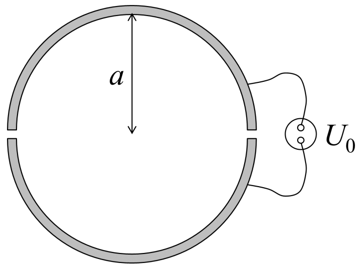
(18th in 19th of October 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A halved conducting tube

[*separation of variables in cylindrical coordinates*]

A long conducting tube of radius a is cut in half along the axis of the tube. The halves are slightly separated and a constant voltage U_0 is applied between them. The figure shows a cross-section of the obtained setup. The wall of the tube is thin, and the gap between the halves is small compared to a.



a) Determine the electric field potential everywhere *inside* the tube as a function of the cylidrical coordinates r and φ (from the horizontal plane) and given parameters a and U_0. The result can be given in the form of an infinite series.

b) Show that, in the *horizontal* symmetry plane inside the tube, the electric field strength points vertically downwards and its size amounts to

$$E(r) = \frac{2U_0 a}{\pi(a^2 - r^2)},$$

where r is the distance from the axis of the tube.

c) Show that, in the *vertical* symmetry plane inside the tube, the electric field strength also points vertically downwards, while its size amounts to

$$E(r) = \frac{2U_0 a}{\pi(a^2 + r^2)}.$$

2. A conducting sphere in a uniform electric field

[*separation of variables in spherical coordinates, Legendre polynomials*]

A conducting sphere of radius a is placed in an external uniform electric field of strength E_0, which distorts the external field.

a) Determine the resulting electric field potential everywhere in space. Interpret the obtained solution qualitatively, i.e., explain the form of both terms.

b) Determine the surface density of the induced charge on the surface of the sphere as a function of the polar angle ϑ from the direction of the external electric field.

c) Determine the electric dipole moment of the induced surface charge. The result can be read directly from the solution under a).

Electromagnetic field: 4th tutorial class

(25th in 26th of October 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A point electric dipole in the center of the conducting sphere

[*separation of variables in spherical coordinates, Legendre polynomials*]

A point electric dipole with electric dipole moment p is placed in the center of an isolated hollow conducting sphere of radius a.

a) Determine the electric field potential everywhere inside the sphere.

b) Show that the electric charge induced on the inner surface of the sphere creates a *uniform* electric field and determine its size.

c) Determine the total dipole moment of the induced charge. What is its direction with respect to the point dipole? Is the result surprising?

2. A point charge above the conducting plane

[*method of images*]

A point electric charge e is located at the distance d above an infinite, grounded conducting plane.

a) Determine the electric field potential everywhere in space. What is the electric field below the plane, i.e., on the other side?

b) Determine the surface density of the induced charge on the plane as a function of the distance r from the point on the plane closest to the point charge. Show that the total induced charge amounts to $-e$. Is there a simple way to obtain this result?

Electromagnetic field: 5th tutorial class

(2nd of November 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A point charge near a conducting sphere

[*method of images*]

A point charge e is located at a distance D from the center of a grounded conducting sphere of radius a.

a) Show that the electric field potential outside the sphere is the same as in the situation where the sphere is replaced by another point charge located on a line between the first point charge and the center of the sphere. Determine the corresponding charge and its location. Determine also the total induced surface charge on the sphere.

b) If the sphere is not grounded, the electric field potential is the same as in the situation where the sphere is replaced by two point charges instead of one. Show that the second point charge should be opposite to the first one and should be placed in the center of a sphere.

Note that the surface of the sphere is an equipotential surface in both cases.

2. A point charge between two perpendicular conducting planes

[*method of images, multipole expansion*]

A point charge e is located between two grounded conducting planes, which are perpendicular to each other, at an equal distance a to each plane.

a) Determine the quadrupole moment of the obtained charge distribution.

b) Determine the electric field potential far away from the point charge.

The electric field potential of the localized charge distribution at $\vec{r}$ is written in the multipole expansion as

$$U(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{r} + \sum_i p_i \frac{r_i}{r^3} + \sum_{ij} Q_{ij}\frac{r_i r_j}{r^5} + \cdots\right),$$

where $p_i = \int \rho(\vec{r}\,')r_i'\,\mathrm{d}^3\vec{r}\,'$ are the components of the dipole moment vector and

$$Q_{ij} = \int \rho(\vec{r}\,')\left[3r_i'r_j' - \delta_{ij}r'^2\right]\mathrm{d}^3\vec{r}\,'$$

are the components of the quadrupole moment tensor, and $\rho(\vec{r}\,')$ is the volume charge density at $\vec{r}\,'$.

3. Electric force on a point charge above the conducting plane

[*electric part of the Maxwell stress tensor*]

A point charge e is located at a distance d above a grounded conducting plane. Using the electric part of the Maxwell stress tensor, determine the electric force on a point charge. Compare the obtained result to the electric force between the point charges e and $-e$ located $2d$ apart.

Electromagnetic field: 6th tutorial class

(8th and 9th of November 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Electric force on the half of a conducting sphere

[*electric part of the Maxwell stress tensor*]

A conducting sphere of radius a is placed in the veritcal uniform electric field of strength E_0. Determine the electric force on the upper half of a sphere. What is the direction of this force?

2. Magnetic field of a current loop

[*magnetic field vector potential, magnetic dipole moment*]

Determine the magnetic field vector potential of a circular loop of radius a carrying the electric field I far away from the loop. Express the result in terms of the distance r from the center of the loop and the angle ϑ with respect to the axis of the loop. Keep only the leading term in r. Show that the result has the form of the magnetic dipole vector potential with the magnetic moment $\pi a^2 I$.

3. Magnetic field of a rotating charged circular plate

[*Biot-Savart law, magnetic dipole moment*]

A horizontal thin circular plate of radius a is uniformly charged with the surface charge density σ. A plate rotates with a constant angular velocity ω around the vertical axis going through the center of the plate.

a) Using Biot-Savart law, determine the magnetic field B on the vertical axis as a function of the distance z from the center of the plate.

b) Show that the magnetic dipole moment of the plate amounts to $p_m = \frac{\pi}{4}\sigma\omega a^4$.

c) Determine the leading term in the Taylor expansion of $B(z)$. Explain why this is the dipole term. Based on its form, determine the dipole moment of the plate and compare it to the result obtained in b).

Electromagnetic field: 7th tutorial class

(15th and 16th of November 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Magnetic force in a coaxial cable

[*magnetic part of the Maxwell stress tensor*]

A long coaxial cable consists of a thin conducting tube of radius a and a thin conducting wire lying on the axis of the tube. A wire is carrying the electric current I, which is returning uniformly distributed across the tube in the opposite direction. Determine the magnetic force per unit length pulling the tube apart.

2. Magnetic force in a long solenoid

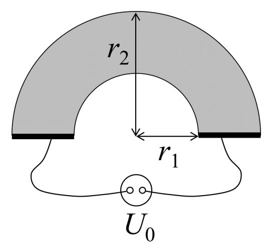
[*magnetic part of the Maxwell stress tensor*]

A long solenoid of length l and the number of turns N, each of radius a, is carrying the electric current I. Determine the magnetic force pulling each turn apart along its circumference.

3. Resistivity of a conducting plate

[*electric field potential in the conductor*]

A thin conducting plate of thickness d is cut from the half of a hollow cylinder of inner radius r_1 and outer radius r_2 made of the material with specific conductivity σ. The straight edges of the plate are covered with ideally conducting electrodes and a voltage U_0 is applied between the electrodes, as shown in the figure. Determine the electric field potential in the plate. Using this, calculate the resistivity of the plate.



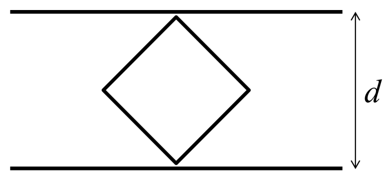
4. Induction in a frame

[*induction, self inductance, mutual inductance*]

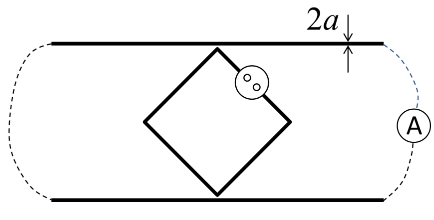
A square-shaped wire frame is placed between two parallel long wires, so that the frame lies in the plane of the wires, whereas the diagonal of the frame is perpendicular to the wires and the corners of the frame just do not touch the wires (see the first figure). The distance between the wires and the diagonal of a frame amount to d.

a) Show that the mutual inductance of the frame and the pair of wires amounts to

$$L_{12} = \frac{2\ln 2}{\pi}\mu_0 d.$$



b) A frame is fed with the alternating electric current of amplitude I_1. Determine the amplitude I_2 of the current induced in the wires when they are connected *far away* (the second figure). In this part of the problem, take into account that the thickness and length of each wire are $2a$ and l, respectively, and that $a \ll d$ and $l \gg d$. Express the final result for I_2/I_1 in terms of d, a and l, and evaluate it numerically for $l/d = d/a = 10$.



Electromagnetic field: 8th tutorial class

(22nd and 23rd of November 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Cabrera's experiment

[*magnetic monopoles, induction*]

In 1982, Blas Cabrera reported an experiment, which allegedly detected a magnetic monopole over a course of 151 days. To detect a magnetic monopole, Cabrera used a circular metallic loop in the superconducting state and measured the electrical current through the loop. Assume that the magnetic monopole of magnetic charge g moves with velocity v along the axis of such a circular loop of radius a and inductivity L.

a) Calculate and plot the time dependence of the magnetic flux through the loop. The monopole magnetic field at $\vec{r}$ relative to the monopole is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0 g}{4\pi} \frac{\vec{r}}{r^3}.$$

b) Calculate and plot the time dependence of the induced electric current in the loop. The generalized Faraday's law for the case of existing magnetic monopoles is written as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{j}_m,$$

where $\vec{j}_m$ is the magnetic charge current density vector.

c) The result obtained in b) shows that the magnetic flux through the loop jumps for $\mu_0 g$ after the magnetic monopole has passed the loop. Show that this value corresponds precisely to two quanta of magnetic flux h/e. Following Dirac, a magnetic flux quantization is written as $\frac{1}{2}\mu_0 g e = h$.

Cabrera's experiment detected a single magnetic monopole. The following similar experiments did not detect any magnetic monopoles.

Electromagnetic field: 9th tutorial class

(29th and 30th of November 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Alternating current in a cylindrical conductor

[*quasistatic aproximation of Maxwell's equations*]

An alternating voltage of amplitude U_0 and angular frequency ω is applied between the ends of a cylindrical conductor made of metal with specific conductivity σ. The radius of a conductor is a, its length is l.

a) Determine the radial dependence of the amplitudes of the electric field strength and the magnetic field in a conductor.

b) Based on the result obtained in a), determine the radial dependence of the amplitude of the electric current density and simplify the result in the limits of small and large ω. Show that, in the limit of large ω, the electric current condenses at the surface of a conductor.

c) Determine the impedance of a conductor. Write the result in terms of the resistance $R_0 = l/(\sigma\pi a^2)$ of a conductor and simplify it in the limits of small and large ω.

Employ the quasistatic aproximation.

2. Solution of the problems from the first written examination

Electromagnetic field: 10th tutorial class

(6th and 7th of December 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Power flux in a coaxial cable and in a cylindrical conductor

[*Poynting theorem*]

Determine the power flux through the cross section or through the outer surface of the following two conductors:

a) a coaxial cable with the voltage U between the core and the shield, which carry opposite constant currents of magnitude I,

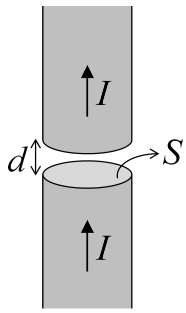
b) a long straight conductor of the cross section S and length l made of metal with the specific conductivity σ carrying the electric current I. Express the final result in terms of the total resistance $R = l/(\sigma S)$ of a conductor.

Interpret the final result in both cases.

2. A split conductor

[*Poynting theorem*]

A long straight cylindrical conductor of the cross section S is split, so that a narrow slit of width d perpendicular to the conductor is obtained (see the figure). A constant electric current I is let through the conductor at the time $t = 0$, which leads to the accumulation of charge on the bottom and top surfaces of the slit.



a) Determine the direction and the magnitude of the electric field strength and of the magnetic field in the slit at a distance r from the axis of the conductor at time t.

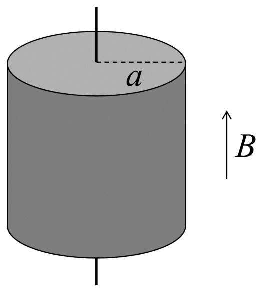
b) Using the Poynting vector, determine the power flux into the slit at time t.

c) Compare the previous result with the time derivative of the electromagnetic field energy in the slit.

In all calculations, neglect the distortion of the fields at the outer edge of the slit. In other words, consider the slit as a plate capacitor. Neglect the resistance of the conductor.

3. Cylindrical version of the Feynman paradox

[*electromagnetic field angular momentum*]



A long straight conductor is uniformly charged with the linear charge density $-\lambda$. A long isolating cylinder of radius a and the moment of inertia j per unit length is mounted around the conductor, so that it can freely rotate around its axis, which coincides with the conductor (see the figure). The surface of the cylinder is covered with the charge of surface density $\lambda/(2\pi a)$, so that the cylinder and the conductor carry the same charge, but of opposite signs. Initially, a uniform magnetic field B is applied along the conductor, which is then gradually turned off.

a) Determine the final angular velocity of the cylinder through the change of the electromagnetic field angular momentum.

b) Determine the angular velocity also directly from the Faraday's law and make sure that you obtain the same result.

c) By changing the linear charge dansity on the conductor to $-\lambda' \neq -\lambda$, while keeping the surface charge density on the cylinder at $\lambda/(2\pi a)$, the determined angular velocity under a) *changes*. Namely, in this case, the electric field outside the cylinder is non-zero, meaning that the same goes for the initial electromagnetic field angular momentum. On the other hand, the angular velocity determined under b) should *not change*. This seeming contradiction is termed the Feynman paradox. How to resolve it?

Electromagnetic field: 11th tutorial class

(13th and 14th of December 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A radially polarized sphere

[*polarization, bound charge*]

A sphere of radius a is polarized so that the polarization vector inside the sphere adopts the form $\vec{P}(\vec{r}) = k\vec{r}$, where k is a known constant. Determine:

a) the volume density of the bound charge inside the sphere, the surface density of the bound charge on the surface of the sphere, and the total charge in the sphere,

b) the electric field strength everywhere in space.

The result under b) shows that the electric field inside the sphere is proportional to the polarization. Interpret this result.

2. A polarized sphere cut in half

[*polarization, bound charge*]

A sphere made of a material with the homogeneous polarization $\vec{P}$ is cut in half, so that the cut is perpendicular to $\vec{P}$ and goes through the center of the sphere. The obtained halves are taken apart, so that the spacing between them is *very small* compared to the radius of the sphere. Determine the electric field strength in the slit between the halves of the sphere. First, solve the problem for the sphere before it is cut, and then consider how the solution is changed in the described situation.

3. A plate made of anisotropic dielectric

[*dielectric constant tensor, boundary conditions for the matter*]

A plate made of anisotropic dielectric in placed between the plates of the plate capacitor of capacitance C_0, so that the plate fills the whole volume of the capacitor. The eigenvalues of the dielectric constant tensor are ε_1, ε_1 and ε_2, and the plate is cut so that the eigenvector corresponding to ε_2 is at an angle φ with respect to the normal of the plates. Determine the capacitance of the filled capacitor.

Electromagnetic field: 12th tutorial class

(20th and 21st of December 2016)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A point electric dipole in the spherical cavity of the dielectric

[*dielectric constant, bound charge, boundary conditions for the matter*]

A large piece of matter with dielectric constant ε contains a spherical cavity of radius a. A point electric dipole with the dipole moment p is placed in the center of the cavity.

a) Determine the electric field potential everywhere in space as a function of the spherical coordinates r and ϑ. Using the obtained result, show that the electric field outside the cavity is that of a electric dipole with the dipole moment $p' = \frac{3p}{2\varepsilon+1}$. The polar angle ϑ is measured from the dipole's direction.

b) Determine the *surface* density of the bound charge on the surface of the spherical cavity as a function of the polar angle ϑ. You can use the result for p' obtained under a).

c) Explain why the *volume* density of the bound charge is zero everywhere in matter.

2. Electromagnetic waves in a cold plasma

[*relation between macroscopic and microscopic quantities*]

Considering the propagation of electromagnetic waves in a cold plasma, one can assume that the constituent ions are almost at rest due to their large mass, while the constituent electrons are almost free, so that they respond instantly to the external fields. In a cold plasma, thermal motion of electrons can be neglected.

a) Using the equation of motion for the free electrons of mass m and charge $-e$, show that the frequency dependence of the dielectric constant of a cold plasma can be written as $\varepsilon(\omega) = 1 - \omega_p^2/\omega^2$, where $\omega_p = \sqrt{ne^2/(m\varepsilon_0)}$ is the plasma frequency and n is the number density of the electrons in plasma.

b) Using the result obtained in a), show that the dispersion relation of the electromagnetic waves propagating in plasma reads $\omega(k) = \sqrt{\omega_p^2 + c_0^2 k^2}$, where c_0 is the speed of the electromagnetic waves in vacuum. Consider also the limiting cases of low and high frequencies.

c) Using the result obtained in b), determine and plot the frequency dependence of the phase and group velocity of the electromagnetic waves in a cold plasma. Compare both velocities to the speed of light in vacuum.

3. Longitudinal electromagnetic waves in matter

[*constitutive relation*]

For the complete description of the behavior of electromagnetic field in matter, we need an additional relation between the individual fields and distributions, such as $\vec{E}$, $\vec{D}$, $\vec{P}$, $\vec{j}$ and ρ. Such a relation is termed a constitutive relation. In ordinary dielectrics, this is the relation between $\vec{D}$ and $\vec{E}$ defining the dielectric constant.

A constitutive relation in a hypothetical matter is written as

$$\frac{\partial \vec{j}}{\partial t} + c_0^2 \nabla \rho = \varepsilon_0 \omega_p^2 \vec{E},$$

where $\vec{j}$ is the density of the electric current, ρ is the volume charge density, while c_0 and ω_p are known constants. Show that *longitudinal* waves can propagate in this matter and determine their dispersion relation. Based on the obtained dispersion relation, can you recognize what kind of matter could that be?

Longitudinal electromagnetic waves are not so frequent as the transverse waves, which are the only possible waves in vacuum.

Electromagnetic field: 13th tutorial class

(3rd and 4th of January 2017)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A wave guide made of two parallel plates

[*electromagnetic wave propagation in a confined geometry*]

We use two large conducting parallel plates separated by a as a wave guide.

a) Denoting the wave propagation direction with z, show that the components E_x, E_y, H_x and H_y of the electric and magnetic field strength can be expressed in terms of the components E_z and H_z. For a total determination of the electromagnetic filed in a wave guide, it is thus sufficient to find the spatial variation of the longitudinal components E_z and H_z. This applies to a wave guide of arbitrary cross-section.

b) Determine the spatial dependence of the longitudinal component E_z for the transverse magnetic (TM) mode of wave propagation where $H_z = 0$, and the spatial dependence of the longitudinal component H_z for the transverse electric (TE) mode of wave propagation where $E_z = 0$. Determine also the dispersion relation in both cases.

c) Show that the wave-guide impedance in the TM mode, which is defined as $Z = E_\perp/H_\parallel$ (where $E_\perp$ is the electric field component perpendicular to the plates and $H_\parallel$ is the magnetic field component parallel to the plates), exhibits the frequency dependence $Z = Z_0\sqrt{1-\omega_0^2/\omega^2}$, where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the resistance of the free space and ω_0 is lowest possible frequency of the wave propagation in the employed mode.

d) Show that the ratio of the *amplitudes* of the transverse and longitudinal electric field components in the TM mode amounts to k/κ, where k is the propagation wave vector and κ is the wavevector describing the transverse electric field variation. The result allows for a simple visualization of the wave propagation along the plates: the wave can be imagined as a periodic bouncing of the wave fronts between the plates.

Electromagnetic field: 14th tutorial class

(10th and 11th of January 2017)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A cylindrical tube as a wave guide

[*electromagnetic wave propagation in a confined geometry*]

A long conducting tube of radius a is used as a wave guide.

a) Determine the spatial dependence of the electric field longitudinal component $E_z(r,\varphi)$ for the transverse magnetic (TM) mode of wave propagation and the spatial dependence of the magnetic field longitudinal component $H_z(r,\varphi)$ for the transverse electric (TE) mode of wave propagation, where the axis of the tube points along z, while r in φ are cylindrical coordinates in the plane perpendicular to z.

b) For each propagation mode, determine the dispersion relation and the minimal possible frequency still allowing the wave propagation.

The tables below summarize the zeros of the Bessel functions and of the derivatives of the Bessel functions, respectively.

k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

k	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

2. Transverse electric and magnetic (TEM) waves in a wave guide

[*electromagnetic wave propagation in a confined geometry*]

In a transverse electric and magnetic (TEM) wave, both electric and magnetic fields, $\vec{E}$ and $\vec{H}$, are perpendicular to the direction of the wave propagation. In a free space, this is the only possible type of wave propagation, while in the wave guides this type of wave propagation is possible only under certain conditions.

a) Show that in the TEM mode the electric field obeys the equation $\nabla \times \vec{E} = i\vec{k} \times \vec{E}$, where $\vec{k}$ is the wave vector, and that the similar equation is valid for $\vec{H}$. Using these two equations, show that the dispersion relation of the TEM wave is linear, $\omega = ck$, where c is the propagation velocity.

b) Show that the amplitudes of both fields obey the Laplace equations $\nabla_\perp^2 \vec{E} = 0$ and $\nabla_\perp^2 \vec{H} = 0$, where $\nabla_\perp$ denotes the gradient operator in the direction perpendicular to the wave propagation direction. These two equations also represent the static limit of the wave equation, i.e., the limit with $\omega = 0$ and $k = 0$. This means that the TEM mode determination is equivalent to solving the static problem for the wave guide of a given cross-section.

c) Using the result under b) explain, why the TEM wave cannot propagate in wave guides with a closed cross-section, while it can propagate, e.g., in a coaxial cable or between the two parallel conducting plates.

3. TEM wave in a coaxial cable

[*electromagnetic wave propagation in a confined geometry*]

A coaxial cable is made of two long conducting tubes of radiuses a and b and thin walls. The space between the walls is filled with a material behaving like plasma with the frequency dependence of the dielectric constant

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2},$$

where ω_p is the plasma frequency. An electromagnetic wave in the TEM mode is introduced into such a wave guide.

a) Determine the dispersion relation of the electromagnetic waves in the wave guide.

b) Determine and plot the frequency dependence of the wave-guide impedance. Explain why the impedance diverges at the frequency ω_p.

4. Radiation of a short dipole antenna

[*radiation approximation*]

We use a straight thin conductor of length l fed with an alternating current $I = I_0 \sin(\omega t)$, which is the same over the whole conductor, as the transmitting antenna. The length of a conductor is short compared to the wave length $\lambda = 2\pi c_0/\omega$ of the transmitted waves, where c_0 is the speed of light in vacuum. Such an antenna is called a Hertzian dipole.

a) Determine the spatial and temporal dependence of the magnetic field $\vec{B}(\vec{r}, t)$ and the electric field strength $\vec{E}(\vec{r}, t)$ in the radiation approximation, i.e., far away from the antenna.

b) Using the Poynting vector, determine the temporal average of the total transmitted power. Express the result in the form ZI_{eff}^2, where Z is a radiation resistance of the antenna and $I_{\text{eff}} = I_0/\sqrt{2}$ is an effective current in the antenna, and determine Z.

Electromagnetic field: 15th tutorial class

(17th and 18th of January 2017)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Radiation of a dipole antenna

[*radiation approximation*]

When a straight thin conductor is used as the transmitting antenna, the current along its length is usually distributed. In particular, a standing current wave is established along the conductor, which for a center-fed conductor can be written as

$$I(z', t) = I_0 \sin\left[k\left(\frac{l}{2} - |z'|\right)\right] \sin(\omega t),$$

where I_0 is the current amplitude, z' the coordinate along the conductor (running from $-l/2$ to $l/2$), and k the wave vector. In case of this particular current distribution, the current drops to zero at the edges of the conductor, just as expected. The shape of the current standing wave is a particularly nice function when the conductor length amounts to an odd multiple of $\lambda/2$, e.g., $l = \lambda/2$, $3\lambda/2 \ldots$

For $l = \lambda/2$ and $l = 3\lambda/2$ determine and plot the spatial distribution of the power flux transmitted by such a dipole antenna. Compare the result for $l = \lambda/2$ to the result for a simple case of a Herzian dipole by plotting both together.

2. Radiation of a small current loop

[*radiation approximation*]

We use a circular current loop of radius a fed with an alternating current $I = I_0 \sin(\omega t)$ as the transmitting antenna. The loop is small compared to the wave length $\lambda = 2\pi c_0/\omega$ of the transmitted waves, where c_0 is the speed of light in vacuum.

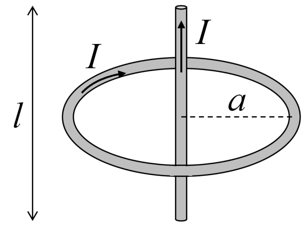
a) Determine the spatial and temporal dependence of the magnetic field $\vec{B}(\vec{r}, t)$ and the electric field strength $\vec{E}(\vec{r}, t)$ in the radiation approximation, i.e., far away from the antenna.

b) Using the Poynting vector, determine the temporal average of the total transmitted power. Express the result in the form ZI_{eff}^2, where Z is a radiation resistance of the antenna and $I_{\text{eff}} = I_0/\sqrt{2}$ is an effective current in the antenna, and determine Z.

3. Radiation of a combined antenna

[*radiation approximation*]

For a transmission of the circularly polarized waves, one can use an antenna shaped like a non-closed horizontal circular loop with vertical ends (see the figure). Let such an antenna be *small* compared to the wave length λ of the transmitted waves. Then it can be treated as a combination of the horizontal circular loop of radius a and the vertical rod of length l passing through the center of the loop (see the figure). Such an antenna is fed with an alternating current $I = I_0 \sin \omega t$.



a) Determine the electric and the magnetic dipole moment of such an antenna as a function of time t.

b) Show that the transmitted waves are elliptically polarized in the arbitrary direction.

c) Given a and a specific wave length λ, how should l be chosen so that the transmitted waves would be circularly polarized?

In the radiation approximation, the magnetic field of an oscillating electric dipole $\vec{p}_e$ at a point $\vec{r}$ can be written as $\vec{B}_e = -\frac{\mu_0}{4\pi c_0 r}\, \hat{e}_r \times \ddot{\vec{p}}_e(t - \frac{r}{c_0})$, while the corresponding expression for a magnetic dipole $\vec{p}_m$ reads $\vec{B}_m = -\frac{\mu_0}{4\pi c_0^2 r}\, \hat{e}_r \times \left[\hat{e}_r \times \ddot{\vec{p}}_m(t - \frac{r}{c_0})\right]$, where c_0 is the speed of light in vacuum and $\hat{e}_r = \vec{r}/r$.