Electromagnetic field: 1st tutorial class

(2nd and 3rd of October 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Electric field of a charged circular plate

[*summing the contributions of the point charges*]

Determine the electric field strength along the axis of a uniformly charged circular plate of radius a as a function of the distance z from the plate. The surface density of the charge on a plate amounts to σ. Simplify the final result in two special cases:

a) near the plate and

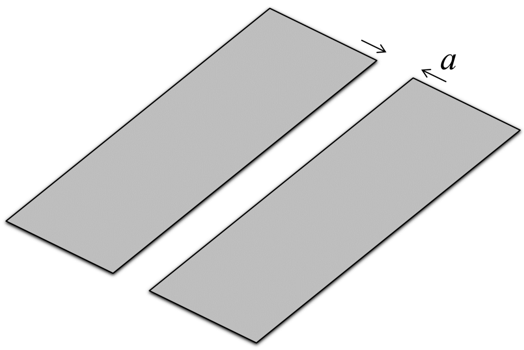
b) far away from the plate.

Compare the obtained results a) with the field of an infinite charged plane and b) with the field of a point charge.

2. Electric field of a charged plane with a slit

[*summing the contributions of the point charges*]

A large flat plate uniformly charged with the surface charge density σ exhibits a straight slit of width a, as shown in the figure.



a) Determine the electric field strength E in the plane perpendicular to the plate, which runs through the middle of the slit, as a function of the distance z from the plate.

b) Simplify the obtained result for $E(z)$ in the limits of small and large z and plot $E(z)$.

Useful series expansion for $x > 0$: $\text{arctg}\left(\frac{1}{x}\right) = \frac{\pi}{2} - x + \cdots$

Electromagnetic field: 2nd tutorial class

(9th and 10th of October 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

0. Explanation and derivation of Fourier transform

[*Fourier transform (FT), inverse FT, FT of gradient and Laplace operator*]

1. Poisson equation for the point charge

[*Fourier transform, Green functions*]

Solve the Poisson equation for the electric field potential of the point charge e,

$$\nabla^2 U(\vec{r}) = -\frac{e}{\varepsilon_0}\delta(\vec{r}),$$

using the Fourier transform.

2. Electric field of the hydrogen atom

[*determination of the charge density from the potential, function derivative*]

The electric field potential of the hydrogen atom in its ground state is given as

$$U(r) = \frac{e}{4\pi\varepsilon_0}\frac{\mathrm{e}^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right),$$

where r is the distance from the atomic nucleus and $\alpha^{-1} = a_B/2$ (a_B is the Bohr radius). Determine the volume charge density leading to such a potential. Interpret the obtained result.

3. Transverse ribbon in a parallel plate capacitor

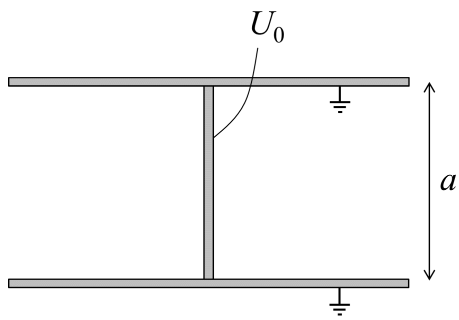
[*separation of variables in cartesian coordinates*]

A long flat conducting ribbon of width a is placed in between the two parallel flat conducting plates separated by a, so that the ribbon is perpendicular to the plates and does not touch them (see figure). The plates are grounded and the ribbon is held at the potential U_0.

a) Determine the electric field potential everywhere inside such a capacitor.

b) Simplify the obtained result for large distances from the ribbon.

c) Calculate the electric field strength in the symmetry plane of the capacitor parallel to the plates. Sum the obtained series.



Electromagnetic field: 3rd tutorial class

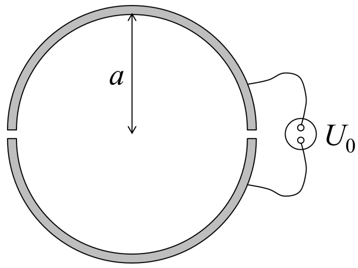
(16th and 17th of October 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A halved conducting tube

[*separation of variables in cylindrical coordinates*]

A long conducting tube of radius a is cut in half along the axis of the tube. The halves are slightly separated and a constant voltage U_0 is applied between them. The figure shows a cross-section of the obtained setup. The wall of the tube is thin, and the gap between the halves is small compared to a.



a) Determine the electric field potential everywhere *inside* the tube as a function of the cylidrical coordinates r and φ (from the horizontal plane) and given parameters a and U_0. The result can be given in the form of an infinite series.

b) Show that, in the *horizontal* symmetry plane inside the tube, the electric field strength points vertically downwards and its size amounts to

$$E(r) = \frac{2U_0 a}{\pi(a^2 - r^2)},$$

where r is the distance from the axis of the tube.

c) Show that, in the *vertical* symmetry plane inside the tube, the electric field strength also points vertically downwards, while its size amounts to

$$E(r) = \frac{2U_0 a}{\pi(a^2 + r^2)}.$$

Electromagnetic field: 4th tutorial class

(23rd and 24th of October 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A conducting sphere in a uniform electric field

[*separation of variables in spherical coordinates, Legendre polynomials*]

A conducting sphere of radius a is placed in an external uniform electric field of strength E_0, which distorts the external field.

a) Determine the resulting electric field potential everywhere in space. Interpret the obtained solution qualitatively, i.e., explain the form of both terms.

b) Determine the surface density of the induced charge on the surface of the sphere as a function of the polar angle ϑ from the direction of the external electric field.

c) Determine the electric dipole moment of the induced surface charge. The result can be read directly from the solution under a).

2. A point electric dipole in the center of the conducting sphere

[*separation of variables in spherical coordinates, Legendre polynomials*]

A point electric dipole with electric dipole moment p is placed in the center of an isolated hollow conducting sphere of radius a.

a) Determine the electric field potential everywhere inside the sphere.

b) Show that the electric charge induced on the inner surface of the sphere creates a *uniform* electric field and determine its size.

c) Determine the total dipole moment of the induced charge. What is its direction with respect to the point dipole? Is the result surprising?

3. A point charge above the conducting plane

[*method of images*]

A point electric charge e is located at the distance d above an infinite, grounded conducting plane.

a) Determine the electric field potential everywhere in space. What is the electric field below the plane, i.e., on the other side?

b) Determine the surface density of the induced charge on the plane as a function of the distance r from the point on the plane closest to the point charge. Show that the total induced charge amounts to $-e$. Is there a simple way to obtain this result?

Electromagnetic field: 5th tutorial class

(30th of October 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Electric force on a point charge above the conducting plane

[*electric part of the Maxwell stress tensor*]

A point charge e is located at a distance d above a grounded conducting plane. Using the electric part of the Maxwell stress tensor, determine the electric force on a point charge. Compare the obtained result to the electric force between the point charges e and $-e$ separated by $2d$.

2. Electric force on the half of a conducting sphere

[*electric part of the Maxwell stress tensor*]

A conducting sphere of radius a is placed in the vertical uniform electric field of strength E_0. Determine the electric force on the upper half of the sphere. What is the direction of this force?

3. A point charge between two perpendicular conducting planes

[*method of images, multipole expansion*]

A point charge e is located between two grounded conducting planes, which are perpendicular to each other, at an equal distance a from each plane.

a) Determine the quadrupole moment of the obtained charge distribution.

b) Determine the electric field potential far away from the point charge.

The electric field potential of the localized charge distribution at $\vec{r}$ is written in the multipole expansion as

$$U(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{r} + \sum_i p_i \frac{r_i}{r^3} + \sum_{ij} Q_{ij}\frac{r_i r_j}{r^5} + \cdots\right),$$

where $p_i = \int \rho(\vec{r}\,')r_i'\,\mathrm{d}^3\vec{r}\,'$ are the components of the dipole moment vector and

$$Q_{ij} = \int \rho(\vec{r}\,')\left[3r_i'r_j' - \delta_{ij}r'^2\right]\mathrm{d}^3\vec{r}\,'$$

are the components of the quadrupole moment tensor, and $\rho(\vec{r}\,')$ is the volume charge density at $\vec{r}\,'$.

4. Magnetic field of a current loop

[*magnetic field vector potential, magnetic dipole moment*]

Determine the magnetic field vector potential of a circular loop of radius a carrying the electric field I far away from the loop. Express the result in terms of the distance r from the center of the loop and the angle ϑ from the axis of the loop. Keep only the leading term in r. Show that the result has the form of the magnetic dipole vector potential with the magnetic moment $\pi a^2 I$.

Electromagnetic field: 6th tutorial class

(5th and 6th of November 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Magnetic field of a rotating charged circular plate
[*Biot-Savart law, magnetic dipole moment*]

A horizontal thin circular plate of radius a is uniformly charged with the surface charge density σ. The plate rotates with a constant angular velocity ω around the vertical axis going through the center of the plate.

a) Using Biot-Savart law, determine the magnetic field B on the vertical axis as a function of the distance z from the center of the plate.

b) Show that the magnetic dipole moment of the plate amounts to $p_m = \frac{\pi}{4}\sigma\omega a^4$.

c) Determine the leading term in the Taylor expansion of $B(z)$. Explain why this is the dipole term. Based on its form, determine the dipole moment of the plate and compare it to the result obtained in b).

Electromagnetic field: 7th tutorial class

(13th and 14th of November 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Magnetic force in a coaxial cable

[*Ampere law, magnetic part of the Maxwell stress tensor*]

A long coaxial cable consists of a thin conducting tube of radius a and a thin conducting wire lying on the axis of the tube. A wire is carrying the electric current I, which is returning uniformly distributed across the tube in the opposite direction. Determine the magnetic force per unit length pulling the tube apart.

2. Magnetic force in a long solenoid

[*Ampere law, magnetic part of the Maxwell stress tensor*]

A toroidal coil with the number of turns N is carrying the electric current I. The radius of each turn is r_1, while the axis of the coil forms a circle of radius r_2 in the horizontal plane.

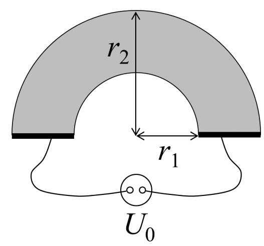
a) Show that the magnetic field in the toroidal coil depends only on the distance r from the vertical axis of the toroid and calculate it. Show that there is no magnetic field outside the coil.

b) Using the Maxwell stress tensor, calculate the magnetic force pulling each turn apart along its circumference for $r_2 \gg r_1$.

3. Resistivity of a conducting plate

[*electric field potential in the conductor*]

A thin conducting plate of thickness d is cut from the half of a hollow cylinder of inner radius r_1 and outer radius r_2 made of the material with specific conductivity σ. The straight edges of the plate are covered with ideally conducting electrodes and a voltage U_0 is applied between the electrodes, as shown in the figure. Determine the electric field potential in the plate. Using this, calculate the resistivity of the plate.



Electromagnetic field: 8th tutorial class

(20th and 21st of November 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

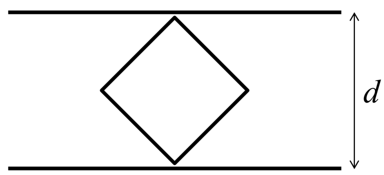
1. Induced current in a frame

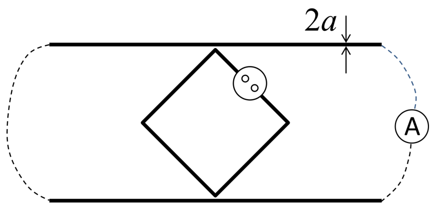
[*magnetic induction, self inductance, mutual inductance*]

A square-shaped wire frame is placed between two parallel long wires, so that the frame lies in the plane of the wires, whereas the diagonal of the frame is perpendicular to the wires and the corners of the frame just do not touch the wires (see the first figure). The distance between the wires and the diagonal of a frame amount to d.

a) Show that the mutual inductance of the frame and the pair of wires amounts to

$$L_{12} = \frac{2\ln 2}{\pi}\mu_0 d.$$





b) A frame is fed with the alternating electric current of amplitude I_1. Determine the amplitude I_2 of the current induced in the wires when they are connected *far away* (the second figure). In this part of the problem, take into account that the thickness and length of each wire are $2a$ and l, respectively, and that $a \ll d$ and $l \gg d$. Express the final result for I_2/I_1 in terms of d, a and l, and evaluate it numerically for $l/d = d/a = 10$.

2. Cabrera's experiment

[*magnetic monopoles, induction*]

In 1982, Blas Cabrera reported an experiment, which allegedly detected a magnetic monopole over a course of 151 days. To detect a magnetic monopole, Cabrera used a circular metallic loop in the superconducting state and measured the electrical current through the loop. Assume that the magnetic monopole of magnetic charge g moves with velocity v along the axis of such a circular loop of radius a and inductivity L.

a) Calculate and plot the time dependence of the magnetic flux through the loop. The monopole magnetic field at $\vec{r}$ relative to the monopole is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0 g}{4\pi}\frac{\vec{r}}{r^3}.$$

b) Calculate and plot the time dependence of the induced electric current in the loop. The generalized Faraday's law for the case of existing magnetic monopoles is written as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{j}_m,$$

where $\vec{j}_m$ is the magnetic charge current density vector.

c) The result obtained in b) shows that the magnetic flux through the loop jumps for $\mu_0 g$ after the magnetic monopole has passed the loop. Show that this value corresponds precisely to two quanta of magnetic flux h/e. Following Dirac, a magnetic flux quantization is written as $\frac{1}{2}\mu_0 ge = h$.

Cabrera's experiment detected a single magnetic monopole. The following similar experiments did not detect any magnetic monopoles.

Electromagnetic field: 9th tutorial class

(27th November 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Alternating current in a wide planar conductor

[*quasistatic aproximation of Maxwell's equations*]

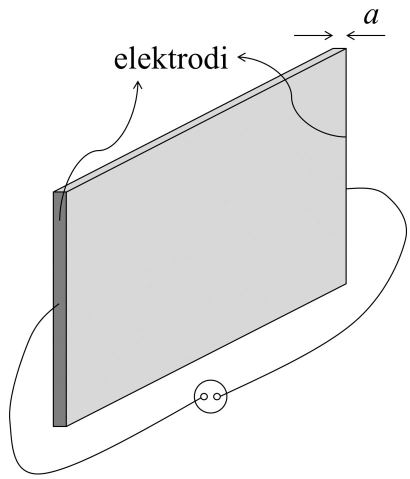
End surfaces of a long wide metal ribbon of thickness a are covered with ideally conducting electrodes. An alternating voltage of angular frequency ω is applied between these surfaces. The thickness of the ribbon is much smaller than its remaining two dimensions, the specific conductivity of the metal is σ.

a) Show that the impedance of the conductor can be written as

$$Z = R_0 \frac{ka/2}{\mathrm{th}\,(ka/2)},$$

where $k = (1+i)\sqrt{\mu_0 \sigma \omega / 2}$ is the complex wave vector, R_0 is the static resistivity of the conductor, and th denotes the hyperbolic tangens.

b) Calculate the factor of increase of the ribbon's electrical resistance at *high* frequencies (due to skin effect) with respect to its static resistance. Calculate also the ribbon's resistance at *low* frequencies.



2. Power flux in a coaxial cable and in a cylindrical conductor

[*Poynting theorem*]

Determine the power flux through the cross section and through the outer surface of the following two conductors:

a) a coaxial cable with the voltage U between the core and the shield, which carry opposite constant currents of magnitude I,

b) a long straight conductor of the cross section S and length l made of metal with the specific conductivity σ carrying the electric current I. Express the final result in terms of the total resistance $R = l/(\sigma S)$ of a conductor.

Interpret the final result in both cases.

2. A split conductor

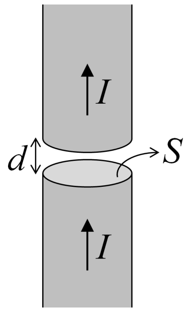
[*Poynting theorem*]

A long straight cylindrical conductor of the cross section S is split, so that a narrow slit of width d perpendicular to the conductor is obtained (see the figure). A constant electric current I is let through the conductor at the time $t = 0$, which leads to the accumulation of charge on the bottom and top surfaces of the slit.

a) Determine the direction and the magnitude of the electric field strength and of the magnetic field in the slit at a distance r from the axis of the conductor at time t.

b) Using the Poynting vector, determine the power flux into the slit at time t.

c) Compare the previous result with the time derivative of the electromagnetic field energy in the slit.



In all calculations, neglect the distortion of the fields at the outer edge of the slit. In other words, consider the slit as a plate capacitor. Neglect the resistance of the conductor.

Electromagnetic field: 10th tutorial class

(4th and 5th of December 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A radially polarized sphere

[*polarization, bound charge*]

A sphere of radius a is polarized so that the polarization vector inside the sphere adopts the form $\vec{P}(\vec{r}) = k\vec{r}$, where k is a known constant. Determine:

a) the volume density of the bound charge inside the sphere, the surface density of the bound charge on the surface of the sphere, and the total charge in the sphere,

b) the electric field strength everywhere in space.

The result under b) shows that the electric field inside the sphere is proportional to the polarization. Interpret this result.

2. A polarized sphere cut in half

[*polarization, bound charge*]

A sphere made of a material with the homogeneous polarization $\vec{P}$ is cut in half, so that the cut is perpendicular to $\vec{P}$ and goes through the center of the sphere. The obtained halves are taken apart, so that the spacing between them is *very small* compared to the radius of the sphere. Determine the electric field strength in the slit between the halves of the sphere. First, solve the problem for the sphere before it is cut, and then consider how the solution is changed in the described situation.

3. A plate made of anisotropic dielectric

[*dielectric constant tensor, boundary conditions for the matter*]

A plate made of anisotropic dielectric in placed between the plates of the plate capacitor of capacitance C_0, so that the plate fills the whole volume of the capacitor. The eigenvalues of the dielectric constant tensor are ε_1, ε_1 and ε_2, and the plate is cut so that the eigenvector corresponding to ε_2 is at an angle φ with respect to the normal of the plates. Determine the capacitance of the filled capacitor.

Electromagnetic field: 11th tutorial class

(11th and 12th of December 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. A point electric dipole in the spherical cavity of the dielectric

[*dielectric constant, bound charge, boundary conditions for the matter*]

A large piece of matter with dielectric constant ε contains a spherical cavity of radius a. A point electric dipole with the dipole moment p is placed in the center of the cavity.

a) Determine the electric field potential everywhere in space as a function of the spherical coordinates r and ϑ. Using the obtained result, show that the electric field outside the cavity is that of an electric dipole with the dipole moment $p' = \frac{3p}{2\varepsilon+1}$. The polar angle ϑ is measured from the dipole's direction.

b) Determine the *surface* density of the bound charge on the surface of the spherical cavity as a function of the polar angle ϑ. You can use the result for p' obtained under a).

c) Explain why the *volume* density of the bound charge is zero everywhere in matter.

2. Electromagnetic waves in a cold plasma

[*relation between macroscopic and microscopic quantities*]

Considering the propagation of electromagnetic waves in a cold plasma, one can assume that the constituent ions are almost at rest due to their large mass, while the constituent electrons are almost free, so that they respond instantly to external fields. In a cold plasma, thermal motion of electrons can be neglected.

a) Using the equation of motion for the free electrons of mass m and charge $-e$, show that the frequency dependence of the dielectric constant of a cold plasma can be written as $\varepsilon(\omega) = 1 - \omega_p^2/\omega^2$, where $\omega_p = \sqrt{ne^2/(m\varepsilon_0)}$ is the plasma frequency and n is the number density of the electrons in plasma.

b) Using the result obtained in a), show that the dispersion relation of the electromagnetic waves propagating in plasma reads $\omega(k) = \sqrt{\omega_p^2 + c_0^2k^2}$, where c_0 is the speed of the electromagnetic waves in vacuum. Consider also the limiting cases of low and high frequencies.

c) Using the result obtained in b), determine and plot the frequency dependence of the phase and group velocity of the electromagnetic waves in a cold plasma. Compare both velocities to the speed of light in vacuum.

Electromagnetic field: 12th tutorial class

(18th and 19th of December 2018)

assistant professor: Martin Klanjšek (01 477 3866, *martin.klanjsek@ijs.si*)

1. Longitudinal electromagnetic waves in matter

[*constitutive relation*]

For the complete description of the behavior of electromagnetic field in matter, we need an additional relation between the individual fields, such as $\vec{E}$, $\vec{D}$, $\vec{P}$, $\vec{j}$ and ρ. Such a relation is termed a constitutive relation. In ordinary dielectrics, this is a relation between $\vec{D}$ and $\vec{E}$ defining the dielectric constant.

A constitutive relation in a hypothetical matter is written as

$$\frac{\partial \vec{j}}{\partial t} + c_0^2 \nabla \rho = \varepsilon_0 \omega_p^2 \vec{E},$$

where $\vec{j}$ is the density of the electric current, ρ is the volume charge density, while c_0 and ω_p are known constants. Show that *longitudinal* waves can propagate in this matter and determine their dispersion relation. Based on the obtained dispersion relation, can you recognize what kind of matter could that be?

Longitudinal electromagnetic waves are not so frequent as the transverse waves, which are the only possible waves in vacuum.

2. A wave guide made of two parallel plates

[*electromagnetic wave propagation in a confined geometry*]

We use two large conducting parallel plates separated by a as a wave guide.

a) Denoting the wave propagation direction with z, show that the components E_x, E_y, H_x and H_y of the electric and magnetic field strength can be expressed in terms of the components E_z and H_z. For a total determination of the electromagnetic filed in a wave guide, it is thus sufficient to find the spatial variation of the longitudinal components E_z and H_z. This applies to a wave guide of arbitrary cross-section.

b) Determine the spatial dependence of the longitudinal component E_z for the transverse magnetic (TM) mode of wave propagation where $H_z = 0$, and the spatial dependence of the longitudinal component H_z for the transverse electric (TE) mode of wave propagation where $E_z = 0$. Determine also the dispersion relation in both cases.

c) Show that the wave-guide impedance in the TM mode, which is defined as $Z = E_\perp / H_\parallel$ (where $E_\perp$ is the electric field component perpendicular to the plates and $H_\parallel$ is the magnetic field component parallel to the plates), exhibits the frequency dependence $Z = Z_0\sqrt{1-\omega_0^2/\omega^2}$, where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the resistance of the free space and ω_0 is lowest possible frequency of the wave propagation in the employed mode.

d) Show that the ratio of the *amplitudes* of the transverse and longitudinal electric field components in the TM mode amounts to k/κ, where k is the propagation wave vector and κ is the wavevector describing the transverse electric field variation. The result allows for a simple visualization of the wave propagation along the plates: the wave can be imagined as a periodic bouncing of the wave fronts between the plates.